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Atlas of Midpoint Algebra

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Summary. This article is a continuation of [5]. We have established a one-to-one correspondence between midpoint algebras and groups with the operator 1/2. In general we shall say that a given midpoint algebra M and a group V are w-assotiated iff w is an atlas from M to V. At the beginning of the paper a few facts which rather belong to [4], [**?**] are proved.

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The articles [3], [7], [2], [1], [6], [4], and [5] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: G denotes a non empty loop structure, x denotes an element of G, M denotes a non empty midpoint algebra structure, p, q, r denote points of M, and w denotes a function from $[:$the carrier of M, the carrier of $M:]$ into the carrier of G.

Let us consider G, x. The functor $2x$ yielding an element of G is defined as follows:

(Def. 1) $2x = x + x$.

Let us consider M, G, w. We say that M, G are associated w.r.t. w if and only if:

(Def. 2) $p \mathbin{@} q = r$ iff $w(p, r) = w(r, q)$.

The following proposition is true

(1) If M, G are associated w.r.t. w, then $p \mathbin{@} p = p$.

We use the following convention: S denotes a non empty set, a, b, b', c, c', d denote elements of S, and w denotes a function from $[:S, S:]$ into the carrier of G.

Let us consider S, G, w. We say that w is an atlas of S, G if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) For all a, x there exists b such that $w(a, b) = x$,

(ii) for all a, b, c such that $w(a, b) = w(a, c)$ holds $b = c$, and

(iii) for all a, b, c holds $w(a, b) + w(b, c) = w(a, c)$.

Let us consider S, G, w, a, x. Let us assume that w is an atlas of S, G. The functor $(a,x).w$ yielding an element of S is defined by:

(Def. 4) $w(a, (a,x).w) = x$.

In the sequel G denotes an add-associative right zeroed right complementable non empty loop structure, x denotes an element of G, and w denotes a function from $[:S, S:]$ into the carrier of G.

The following propositions are true:

(2) $2(0_G) = 0_G$.

(4)[1] If w is an atlas of S, G, then $w(a, a) = 0_G$.

(5) If w is an atlas of S, G and $w(a, b) = 0_G$, then $a = b$.

(6) If w is an atlas of S, G, then $w(a, b) = -w(b, a)$.

(7) If w is an atlas of S, G and $w(a, b) = w(c, d)$, then $w(b, a) = w(d, c)$.

(8) If w is an atlas of S, G, then for all b, x there exists a such that $w(a, b) = x$.

(9) If w is an atlas of S, G and $w(b, a) = w(c, a)$, then $b = c$.

(10) Let w be a function from $[:$the carrier of M, the carrier of $M:]$ into the carrier of G. Suppose w is an atlas of the carrier of M, G and M, G are associated w.r.t. w. Then $p\,@\,q = q\,@\,p$.

(11) Let w be a function from $[:$the carrier of M, the carrier of $M:]$ into the carrier of G. Suppose w is an atlas of the carrier of M, G and M, G are associated w.r.t. w. Then there exists r such that $r\,@\,p = q$.

In the sequel G is an add-associative right zeroed right complementable Abelian non empty loop structure and x is an element of G.

The following propositions are true:

(13)[2] Let G be an add-associative Abelian non empty loop structure and x, y, z, t be elements of G. Then $(x+y)+(z+t) = x+z+(y+t)$.

(14) For every add-associative Abelian non empty loop structure G and for all elements x, y of G holds $2(x+y) = 2x+2y$.

(15) $2(-x) = -2x$.

(16) Let w be a function from $[:$the carrier of M, the carrier of $M:]$ into the carrier of G. Suppose w is an atlas of the carrier of M, G and M, G are associated w.r.t. w. Let a, b, c, d be points of M. Then $a\,@\,b = c\,@\,d$ if and only if $w(a, d) = w(c, b)$.

In the sequel w denotes a function from $[:S, S:]$ into the carrier of G.

One can prove the following proposition

(17) If w is an atlas of S, G, then for all a, b, b', c, c' such that $w(a, b) = w(b, c)$ and $w(a, b') = w(b', c')$ holds $w(c, c') = 2w(b, b')$.

We use the following convention: M denotes a midpoint algebra and p, q, r, s denote points of M.

Let us consider M. Observe that vectgroup$M$ is Abelian, add-associative, right zeroed, and right complementable.

We now state the proposition

(18)(i) For every set a holds a is an element of vectgroup$M$ iff a is a vector of M,

(ii) $0_{\mathrm{vectgroup}M} = \mathrm{I}_M$, and

(iii) for all elements a, b of vectgroup$M$ and for all vectors x, y of M such that $a = x$ and $b = y$ holds $a+b = x+y$.

Let I_1 be a non empty loop structure. We say that I_1 is midpoint operator if and only if the conditions (Def. 5) are satisfied.

(Def. 5)(i) For every element a of I_1 there exists an element x of I_1 such that $2x = a$, and

(ii) for every element a of I_1 such that $2a = 0_{(I_1)}$ holds $a = 0_{(I_1)}$.

[1] The proposition (3) has been removed.
[2] The proposition (12) has been removed.

Let us observe that every non empty loop structure which is midpoint operator is also Fanoian.

Let us mention that there exists a non empty loop structure which is strict, midpoint operator, add-associative, right zeroed, right complementable, and Abelian.

In the sequel G denotes a midpoint operator add-associative right zeroed right complementable Abelian non empty loop structure and x, y denote elements of G.

We now state two propositions:

(19) Let G be a Fanoian add-associative right zeroed right complementable non empty loop structure and x be an element of G. If $x = -x$, then $x = 0_G$.

(20) Let G be a Fanoian add-associative right zeroed right complementable Abelian non empty loop structure and x, y be elements of G. If $2x = 2y$, then $x = y$.

Let G be a midpoint operator add-associative right zeroed right complementable Abelian non empty loop structure and let x be an element of G. The functor $\frac{1}{2}x$ yielding an element of G is defined as follows:

(Def. 6) $2\frac{1}{2}x = x$.

The following propositions are true:

(21) $\frac{1}{2}(0_G) = 0_G$ and $\frac{1}{2}(x+y) = \frac{1}{2}x + \frac{1}{2}y$ and if $\frac{1}{2}x = \frac{1}{2}y$, then $x = y$ and $\frac{1}{2}2x = x$.

(22) Let M be a non empty midpoint algebra structure and w be a function from $[:$the carrier of M, the carrier of $M:]$ into the carrier of G. Suppose w is an atlas of the carrier of M, G and M, G are associated w.r.t. w. Let a, b, c, d be points of M. Then $(a\,^{@}\,b)\,^{@}\,(c\,^{@}\,d) = a\,^{@}\,c\,^{@}\,(b\,^{@}\,d)$.

(23) Let M be a non empty midpoint algebra structure and w be a function from $[:$the carrier of M, the carrier of $M:]$ into the carrier of G. Suppose w is an atlas of the carrier of M, G and M, G are associated w.r.t. w. Then M is a midpoint algebra.

Let us consider M. Observe that vectgroup$M$ is midpoint operator.

Let us consider M, p, q. The functor q^p yielding an element of vectgroup$M$ is defined as follows:

(Def. 7) $q^p = \overrightarrow{[p,q]}$.

Let us consider M. The functor vect$M$ yielding a function from $[:$the carrier of M, the carrier of $M:]$ into the carrier of vectgroup$M$ is defined by:

(Def. 8) $(\mathrm{vect}M)(p, q) = \overrightarrow{[p,q]}$.

One can prove the following propositions:

(24) vect$M$ is an atlas of the carrier of M, vectgroup$M$.

(25) $\overrightarrow{[p,q]} = \overrightarrow{[r,s]}$ iff $p\,^{@}\,s = q\,^{@}\,r$.

(26) $p\,^{@}\,q = r$ iff $\overrightarrow{[p,r]} = \overrightarrow{[r,q]}$.

(27) M, vectgroup$M$ are associated w.r.t. vect$M$.

In the sequel w is a function from $[:S, S:]$ into the carrier of G.

Let us consider S, G, w. Let us assume that w is an atlas of S, G. The functor $^{@}w$ yielding a binary operation on S is defined as follows:

(Def. 9) $w(a, (^{@}w)(a, b)) = w((^{@}w)(a, b), b)$.

We now state the proposition

(28) If w is an atlas of S, G, then for all a, b, c holds $(^{@}w)(a, b) = c$ iff $w(a, c) = w(c, b)$.

Let D be a non empty set and let M be a binary operation on D. Observe that $\langle D, M \rangle$ is non empty.

Let us consider S, G, w. The functor Atlas w yields a function from [: the carrier of $\langle S, {}^{@}w \rangle$, the carrier of $\langle S, {}^{@}w \rangle$:] into the carrier of G and is defined by:

(Def. 10) Atlas $w = w$.

We now state the proposition

(32)[3] If w is an atlas of S, G, then $\langle S, {}^{@}w \rangle$, G are associated w.r.t. Atlas w.

Let us consider S, G, w. Let us assume that w is an atlas of S, G. The functor $\mathrm{MidSp}(w)$ yields a strict midpoint algebra and is defined by:

(Def. 11) $\mathrm{MidSp}(w) = \langle S, {}^{@}w \rangle$.

We adopt the following rules: M is a non empty midpoint algebra structure, w is a function from [: the carrier of M, the carrier of M :] into the carrier of G, and a, b, b_1, b_2, c are points of M.

Next we state the proposition

(33) M is a midpoint algebra if and only if there exists G and there exists w such that w is an atlas of the carrier of M, G and M, G are associated w.r.t. w.

Let M be a non empty midpoint algebra structure. We introduce atlas structures over M which are systems

⟨ an algebra, a function ⟩,

where the algebra is a non empty loop structure and the function is a function from [: the carrier of M, the carrier of M :] into the carrier of the algebra.

Let M be a non empty midpoint algebra structure and let I_1 be an atlas structure over M. We say that I_1 is atlas-like if and only if the conditions (Def. 12) are satisfied.

(Def. 12)(i) The algebra of I_1 is midpoint operator, add-associative, right zeroed, right complementable, and Abelian,

(ii) M, the algebra of I_1 are associated w.r.t. the function of I_1, and

(iii) the function of I_1 is an atlas of the carrier of M, the algebra of I_1.

Let M be a midpoint algebra. One can check that there exists an atlas structure over M which is atlas-like.

Let M be a non empty midpoint algebra. An atlas of M is an atlas-like atlas structure over M.

Let M be a non empty midpoint algebra structure and let W be an atlas structure over M. A vector of W is an element of the algebra of W.

Let M be a midpoint algebra, let W be an atlas structure over M, and let a, b be points of M. The functor $W(a, b)$ yields an element of the algebra of W and is defined by:

(Def. 13) $W(a, b) = (\text{the function of } W)(a, b)$.

Let M be a midpoint algebra, let W be an atlas structure over M, let a be a point of M, and let x be a vector of W. The functor $(a, x).W$ yields a point of M and is defined as follows:

(Def. 14) $(a, x).W = (a, x).\text{the function of } W$.

Let M be a midpoint algebra and let W be an atlas of M. The functor 0_W yields a vector of W and is defined as follows:

(Def. 15) $0_W = 0_{\text{the algebra of } W}$.

We now state two propositions:

(34) Suppose w is an atlas of the carrier of M, G and M, G are associated w.r.t. w. Then $a \mathbin{@} c = b_1 \mathbin{@} b_2$ if and only if $w(a, c) = w(a, b_1) + w(a, b_2)$.

[3] The propositions (29)–(31) have been removed.

(35) Suppose w is an atlas of the carrier of M, G and M, G are associated w.r.t. w. Then $a\,^{@}\,c = b$ if and only if $w(a, c) = 2w(a, b)$.

For simplicity, we adopt the following rules: M denotes a midpoint algebra, W denotes an atlas of M, a, b, b_1, b_2, c, d denote points of M, and x denotes a vector of W.

We now state several propositions:

(36) $a\,^{@}\,c = b_1\,^{@}\,b_2$ iff $W(a, c) = W(a, b_1) + W(a, b_2)$.

(37) $a\,^{@}\,c = b$ iff $W(a, c) = 2W(a, b)$.

(38)(i) For all a, x there exists b such that $W(a, b) = x$,

(ii) for all a, b, c such that $W(a, b) = W(a, c)$ holds $b = c$, and

(iii) for all a, b, c holds $W(a, b) + W(b, c) = W(a, c)$.

(39) $W(a, a) = 0_W$ and if $W(a, b) = 0_W$, then $a = b$ and $W(a, b) = -W(b, a)$ and if $W(a, b) = W(c, d)$, then $W(b, a) = W(d, c)$ and for all b, x there exists a such that $W(a, b) = x$ and if $W(b, a) = W(c, a)$, then $b = c$ and $a\,^{@}\,b = c$ iff $W(a, c) = W(c, b)$ and $a\,^{@}\,b = c\,^{@}\,d$ iff $W(a, d) = W(c, b)$ and $W(a, b) = x$ iff $(a, x).W = b$.

(40) $(a, 0_W).W = a$.

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