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Sequences in Metric Spaces

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Summary. Sequences in metric spaces are defined. The article contains definitions of bounded, convergent, Cauchy sequences. The subsequences are introduced too. Some theorems concerning sequences are proved.

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The articles [15], [5], [17], [1], [16], [12], [18], [3], [4], [7], [8], [11], [9], [10], [2], [6], [13], and [14] provide the notation and terminology for this paper.

For simplicity, we follow the rules: X is a metric space, x, y, z are elements of X, A is a non empty set, a is an element of A, G is a function from $[:A, A:]$ into $\mathbb{R}$, n, m are natural numbers, and r is a real number.

Next we state several propositions:

(1) $|\rho(x,z) - \rho(y,z)| \leq \rho(x,y)$.

(2) If G is a metric of A, then for all elements a, b of A holds $0 \leq G(a, b)$.

(3) G is a metric of A iff G is Reflexive, discernible, symmetric, and triangle.

(4) For every strict non empty metric space X holds the distance of X is Reflexive, discernible, symmetric, and triangle.

(5) G is a metric of A if and only if the following conditions are satisfied:
(i) G is Reflexive and discernible, and
(ii) for all elements a, b, c of A holds $G(b, c) \leq G(a, b) + G(a, c)$.

Let us consider A and let us consider G. The functor $\widetilde{G}_A$ yielding a function from $[:A, A:]$ into $\mathbb{R}$ is defined as follows:

(Def. 4)[1] For all elements a, b of A holds $\widetilde{G}_A(a, b) = \frac{G(a,b)}{1+G(a,b)}$.

The following proposition is true

(6) If G is a metric of A, then $\widetilde{G}_A$ is a metric of A.

For simplicity, we use the following convention: X is a non empty metric space, x, y are elements of X, V is a subset of X, S, S_1, T are sequences of X, and F is a function from $\mathbb{N}$ into the carrier of X.

Next we state two propositions:

[1] The definitions (Def. 1)–(Def. 3) have been removed.

(8)[2] F is a sequence of X iff for every a such that $a \in \mathbb{N}$ holds $F(a)$ is an element of X.

(10)[3] For every x there exists S such that $\operatorname{rng} S = \{x\}$.

Let us consider X, let us consider S, and let us consider x. We say that S is convergent to x if and only if:

(Def. 8)[4] For every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\rho(S(n), x) < r$.

Let us consider X and let V be a subset of X. Let us observe that V is bounded if and only if:

(Def. 10)[5] There exist r, x such that $0 < r$ and $V \subseteq \mathrm{Ball}(x, r)$.

Let us consider X and let us consider S. We say that S is bounded if and only if:

(Def. 11) There exist r, x such that $0 < r$ and $\operatorname{rng} S \subseteq \mathrm{Ball}(x, r)$.

Let us consider X, let us consider V, and let us consider S. We say that V contains almost all sequence S if and only if:

(Def. 12) There exists m such that for every n such that $m \leq n$ holds $S(n) \in V$.

The following propositions are true:

(20)[6] S is bounded iff there exist r, x such that $0 < r$ and for every n holds $S(n) \in \mathrm{Ball}(x, r)$.

(21) If S is convergent to x, then S is convergent.

(22) If S is convergent, then there exists x such that S is convergent to x.

Let us consider X, let us consider S, and let us consider x. The functor $\rho(S, x)$ yields a sequence of real numbers and is defined by:

(Def. 14)[7] For every n holds $(\rho(S, x))(n) = \rho(S(n), x)$.

Let us consider X, let us consider S, and let us consider T. The functor $\rho(S, T)$ yielding a sequence of real numbers is defined as follows:

(Def. 15) For every n holds $(\rho(S, T))(n) = \rho(S(n), T(n))$.

Let us consider X and let us consider S. Let us assume that S is convergent. The functor $\lim S$ yields an element of X and is defined as follows:

(Def. 16) For every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\rho(S(n), \lim S) < r$.

One can prove the following propositions:

(26)[8] If S is convergent to x, then $\lim S = x$.

(27) S is convergent to x iff S is convergent and $\lim S = x$.

(28) If S is convergent, then there exists x such that S is convergent to x and $\lim S = x$.

(29) S is convergent to x iff $\rho(S, x)$ is convergent and $\lim \rho(S, x) = 0$.

[2] The proposition (7) has been removed.
[3] The proposition (9) has been removed.
[4] The definitions (Def. 5)–(Def. 7) have been removed.
[5] The definition (Def. 9) has been removed.
[6] The propositions (11)–(19) have been removed.
[7] The definition (Def. 13) has been removed.
[8] The propositions (23)–(25) have been removed.

(30) If S is convergent to x, then for every r such that $0 < r$ holds $\mathrm{Ball}(x,r)$ contains almost all sequence S.

(31) Suppose that for every r such that $0 < r$ holds $\mathrm{Ball}(x,r)$ contains almost all sequence S. Let given V. Suppose $x \in V$ and $V \in$ the open set family of X. Then V contains almost all sequence S.

(32) Suppose that for every V such that $x \in V$ and $V \in$ the open set family of X holds V contains almost all sequence S. Then S is convergent to x.

(33) S is convergent to x iff for every r such that $0 < r$ holds $\mathrm{Ball}(x,r)$ contains almost all sequence S.

(34) S is convergent to x if and only if for every V such that $x \in V$ and $V \in$ the open set family of X holds V contains almost all sequence S.

(35) The following statements are equivalent

(i) for every r such that $0 < r$ holds $\mathrm{Ball}(x,r)$ contains almost all sequence S,

(ii) for every V such that $x \in V$ and $V \in$ the open set family of X holds V contains almost all sequence S.

(36) If S is convergent and T is convergent, then $\rho(\lim S, \lim T) = \lim \rho(S,T)$.

(37) If S is convergent to x and convergent to y, then $x = y$.

(38) If S is constant, then S is convergent.

(39) If S is convergent to x and S_1 is a subsequence of S, then S_1 is convergent to x.

(40) If S is Cauchy and S_1 is a subsequence of S, then S_1 is Cauchy.

(42)[9] If S is constant, then S is Cauchy.

(43) If S is convergent, then S is bounded.

(44) If S is Cauchy, then S is bounded.

Let M be a non empty metric space. One can verify the following observations:

* every sequence of M which is constant is also convergent,
* every sequence of M which is convergent is also Cauchy, and
* every sequence of M which is Cauchy is also bounded.

Let M be a non empty metric space. One can verify that there exists a sequence of M which is constant, convergent, Cauchy, and bounded.

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