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Metrics in the Cartesian Product — Part II

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Summary. A continuation of [5]. It deals with the method of creation of the distance in the Cartesian product of metric spaces. The distance between two points belonging to Cartesian product of metric spaces has been defined as square root of the sum of squares of distances of appropriate coordinates (or projections) of these points. It is shown that product of metric spaces with such a distance is a metric space. Examples of metric spaces defined in this way are given.

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The articles [2], [9], [7], [3], [1], [4], [6], and [8] provide the notation and terminology for this paper.

1. Metrics in the Cartesian product of two sets

We adopt the following rules: X, Y are non empty metric spaces, x_1, y_1, z_1 are elements of X, and x_2, y_2, z_2 are elements of Y.

Let us consider X, Y. The functor $\rho^{[:X,Y:]}$ yields a function from $[:[:$the carrier of X, the carrier of $Y:]$, $[:$the carrier of X, the carrier of $Y:]:]$ into $\mathbb{R}$ and is defined by the condition (Def. 1).

(Def. 1) Let x_1, y_1 be elements of X, x_2, y_2 be elements of Y, and x, y be elements of $[:$the carrier of X, the carrier of $Y:]$. If $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$, then $\rho^{[:X,Y:]}(x, y) = \sqrt{(\rho(x_1,y_1))^2 + (\rho(x_2,y_2))^2}$.

The following propositions are true:

(2)[1] For all elements a, b of $\mathbb{R}$ such that $0 \leq a$ and $0 \leq b$ holds $\sqrt{a+b} = 0$ iff $a = 0$ and $b = 0$.

(3) Let x, y be elements of $[:$the carrier of X, the carrier of $Y:]$. If $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$, then $\rho^{[:X,Y:]}(x, y) = 0$ iff $x = y$.

(4) For all elements x, y of $[:$the carrier of X, the carrier of $Y:]$ such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $\rho^{[:X,Y:]}(x, y) = \rho^{[:X,Y:]}(y, x)$.

(5) For all elements a, b, c, d of $\mathbb{R}$ such that $0 \leq a$ and $0 \leq b$ and $0 \leq c$ and $0 \leq d$ holds $\sqrt{(a+c)^2 + (b+d)^2} \leq \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2}$.

(6) Let x, y, z be elements of $[:$the carrier of X, the carrier of $Y:]$. If $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ and $z = \langle z_1, z_2 \rangle$, then $\rho^{[:X,Y:]}(x, z) \leq \rho^{[:X,Y:]}(x, y) + \rho^{[:X,Y:]}(y, z)$.

Let us consider X, Y and let x, y be elements of $[:$the carrier of X, the carrier of $Y:]$. The functor $\rho^2(x,y)$ yielding a real number is defined as follows:

[1] The proposition (1) has been removed.

(Def. 2) $\rho^{2}(x,y) = \rho^{[:X,Y:]}(x, y)$.

Let us consider X, Y. The functor $[:X,Y:]$ yields a strict non empty metric space and is defined as follows:

(Def. 3) $[:X,Y:] = \langle [:\text{the carrier of } X, \text{ the carrier of } Y:], \rho^{[:X,Y:]} \rangle$.

The following proposition is true

(8)[2] $\langle [:\text{the carrier of } X, \text{ the carrier of } Y:], \rho^{[:X,Y:]} \rangle$ is a metric space.

2. Metrics in the Cartesian product of three sets

In the sequel Z denotes a non empty metric space and x_3, y_3, z_3 denote elements of Z.

Let us consider X, Y, Z. The functor $\rho^{[:X,Y,Z:]}$ yielding a function from $[:[:\text{the carrier of } X, \text{ the carrier of } Y, \text{ the carrier of } Z:], [:\text{the carrier of } X, \text{ the carrier of } Y, \text{ the carrier of } Z:]:]$ into $\mathbb{R}$ is defined by the condition (Def. 4).

(Def. 4) Let x_1, y_1 be elements of X, x_2, y_2 be elements of Y, x_3, y_3 be elements of Z, and x, y be elements of $[:\text{the carrier of } X, \text{ the carrier of } Y, \text{ the carrier of } Z:]$. If $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$, then $\rho^{[:X,Y,Z:]}(x, y) = \sqrt{(\rho(x_1,y_1))^{2} + (\rho(x_2,y_2))^{2} + (\rho(x_3,y_3))^{2}}$.

Next we state several propositions:

(10)[3] Let x, y be elements of $[:\text{the carrier of } X, \text{ the carrier of } Y, \text{ the carrier of } Z:]$. If $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$, then $\rho^{[:X,Y,Z:]}(x, y) = 0$ iff $x = y$.

(11) Let x, y be elements of $[:\text{the carrier of } X, \text{ the carrier of } Y, \text{ the carrier of } Z:]$. If $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$, then $\rho^{[:X,Y,Z:]}(x, y) = \rho^{[:X,Y,Z:]}(y, x)$.

(12) For all elements a, b, c of $\mathbb{R}$ holds $(a+b+c)^{2} = a^{2}+b^{2}+c^{2}+(2\cdot a\cdot b+2\cdot a\cdot c+2\cdot b\cdot c)$.

(13) For all elements a, b, c, d, e, f of $\mathbb{R}$ holds $2\cdot(a\cdot d)\cdot(c\cdot b)+2\cdot(a\cdot f)\cdot(e\cdot c)+2\cdot(b\cdot f)\cdot(e\cdot d) \leq (a\cdot d)^{2}+(c\cdot b)^{2}+(a\cdot f)^{2}+(e\cdot c)^{2}+(b\cdot f)^{2}+(e\cdot d)^{2}$.

(14) Let a, b, c, d, e, f be elements of $\mathbb{R}$. Then $a^{2}\cdot d^{2}+(a^{2}\cdot f^{2}+c^{2}\cdot b^{2})+e^{2}\cdot c^{2}+b^{2}\cdot f^{2}+e^{2}\cdot d^{2}+e^{2}\cdot f^{2}+b^{2}\cdot d^{2}+a^{2}\cdot c^{2} = (a^{2}+b^{2}+e^{2})\cdot(c^{2}+d^{2}+f^{2})$.

(15) For all elements a, b, c, d, e, f of $\mathbb{R}$ holds $(a\cdot c+b\cdot d+e\cdot f)^{2} \leq (a^{2}+b^{2}+e^{2})\cdot(c^{2}+d^{2}+f^{2})$.

(16) Let x, y, z be elements of $[:\text{the carrier of } X, \text{ the carrier of } Y, \text{ the carrier of } Z:]$. If $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ and $z = \langle z_1, z_2, z_3 \rangle$, then $\rho^{[:X,Y,Z:]}(x, z) \leq \rho^{[:X,Y,Z:]}(x, y) + \rho^{[:X,Y,Z:]}(y, z)$.

Let us consider X, Y, Z and let x, y be elements of $[:\text{the carrier of } X, \text{ the carrier of } Y, \text{ the carrier of } Z:]$. The functor $\rho^{3}(x,y)$ yields a real number and is defined by:

(Def. 5) $\rho^{3}(x,y) = \rho^{[:X,Y,Z:]}(x, y)$.

Let us consider X, Y, Z. The functor $[:X,Y:]$ yields a strict non empty metric space and is defined by:

(Def. 6) $[:X,Y:] = \langle [:\text{the carrier of } X, \text{ the carrier of } Y, \text{ the carrier of } Z:], \rho^{[:X,Y,Z:]} \rangle$.

One can prove the following proposition

(18)[4] $\langle [:\text{the carrier of } X, \text{ the carrier of } Y, \text{ the carrier of } Z:], \rho^{[:X,Y,Z:]} \rangle$ is a metric space.

[2] The proposition (7) has been removed.
[3] The proposition (9) has been removed.
[4] The proposition (17) has been removed.

In the sequel $x_1, x_2, y_1, y_2, z_1, z_2$ denote elements of $\mathbb{R}$.
The function $\rho^{[:\mathbb{R},\mathbb{R}:]}$ from $[:[:\mathbb{R},\mathbb{R}:],[:\mathbb{R},\mathbb{R}:]:]$ into $\mathbb{R}$ is defined as follows:

(Def. 7) For all elements x_1, y_1, x_2, y_2 of $\mathbb{R}$ and for all elements x, y of $[:\mathbb{R},\mathbb{R}:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R}:]}(x, y) = \rho_{\mathbb{R}}(x_1, y_1) + \rho_{\mathbb{R}}(x_2, y_2)$.

One can prove the following propositions:

(19) Let x_1, x_2, y_1, y_2 be elements of $\mathbb{R}$ and x, y be elements of $[:\mathbb{R},\mathbb{R}:]$. If $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$, then $\rho^{[:\mathbb{R},\mathbb{R}:]}(x, y) = 0$ iff $x = y$.

(20) For all elements x, y of $[:\mathbb{R},\mathbb{R}:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R}:]}(x, y) = \rho^{[:\mathbb{R},\mathbb{R}:]}(y, x)$.

(21) For all elements x, y, z of $[:\mathbb{R},\mathbb{R}:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ and $z = \langle z_1, z_2\rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R}:]}(x, z) \leq \rho^{[:\mathbb{R},\mathbb{R}:]}(x, y) + \rho^{[:\mathbb{R},\mathbb{R}:]}(y, z)$.

The strict non empty metric space $[:\mathbb{R}_{\mathrm{M}},\mathbb{R}_{\mathrm{M}}:]$ is defined as follows:

(Def. 8) $[:\mathbb{R}_{\mathrm{M}},\mathbb{R}_{\mathrm{M}}:] = \langle [:\mathbb{R},\mathbb{R}:], \rho^{[:\mathbb{R},\mathbb{R}:]}\rangle$.

The function $\rho^{\mathbb{R}^2}$ from $[:[:\mathbb{R},\mathbb{R}:],[:\mathbb{R},\mathbb{R}:]:]$ into $\mathbb{R}$ is defined by the condition (Def. 9).

(Def. 9) Let x_1, y_1, x_2, y_2 be elements of $\mathbb{R}$ and x, y be elements of $[:\mathbb{R},\mathbb{R}:]$. If $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$, then $\rho^{\mathbb{R}^2}(x, y) = \sqrt{\rho_{\mathbb{R}}(x_1, y_1)^{\mathbf{2}} + \rho_{\mathbb{R}}(x_2, y_2)^{\mathbf{2}}}$.

One can prove the following propositions:

(22) Let x_1, x_2, y_1, y_2 be elements of $\mathbb{R}$ and x, y be elements of $[:\mathbb{R},\mathbb{R}:]$. If $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$, then $\rho^{\mathbb{R}^2}(x, y) = 0$ iff $x = y$.

(23) For all elements x, y of $[:\mathbb{R},\mathbb{R}:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds $\rho^{\mathbb{R}^2}(x, y) = \rho^{\mathbb{R}^2}(y, x)$.

(24) For all elements x, y, z of $[:\mathbb{R},\mathbb{R}:]$ such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ and $z = \langle z_1, z_2\rangle$ holds $\rho^{\mathbb{R}^2}(x, z) \leq \rho^{\mathbb{R}^2}(x, y) + \rho^{\mathbb{R}^2}(y, z)$.

The strict non empty metric space the Euclidean plane is defined by:

(Def. 10) The Euclidean plane $= \langle [:\mathbb{R},\mathbb{R}:], \rho^{\mathbb{R}^2}\rangle$.

In the sequel x_3, y_3, z_3 are elements of $\mathbb{R}$.
The function $\rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}$ from $[:[:\mathbb{R},\mathbb{R},\mathbb{R}:],[:\mathbb{R},\mathbb{R},\mathbb{R}:]:]$ into $\mathbb{R}$ is defined by the condition (Def. 11).

(Def. 11) Let $x_1, y_1, x_2, y_2, x_3, y_3$ be elements of $\mathbb{R}$ and x, y be elements of $[:\mathbb{R},\mathbb{R},\mathbb{R}:]$. If $x = \langle x_1, x_2, x_3\rangle$ and $y = \langle y_1, y_2, y_3\rangle$, then $\rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(x, y) = \rho_{\mathbb{R}}(x_1, y_1) + \rho_{\mathbb{R}}(x_2, y_2) + \rho_{\mathbb{R}}(x_3, y_3)$.

Next we state three propositions:

(25) Let $x_1, x_2, y_1, y_2, x_3, y_3$ be elements of $\mathbb{R}$ and x, y be elements of $[:\mathbb{R},\mathbb{R},\mathbb{R}:]$. If $x = \langle x_1, x_2, x_3\rangle$ and $y = \langle y_1, y_2, y_3\rangle$, then $\rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(x, y) = 0$ iff $x = y$.

(26) For all elements x, y of $[:\mathbb{R},\mathbb{R},\mathbb{R}:]$ such that $x = \langle x_1, x_2, x_3\rangle$ and $y = \langle y_1, y_2, y_3\rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(x, y) = \rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(y, x)$.

(27) For all elements x, y, z of $[:\mathbb{R},\mathbb{R},\mathbb{R}:]$ such that $x = \langle x_1, x_2, x_3\rangle$ and $y = \langle y_1, y_2, y_3\rangle$ and $z = \langle z_1, z_2, z_3\rangle$ holds $\rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(x, z) \leq \rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(x, y) + \rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}(y, z)$.

The strict non empty metric space $[:\mathbb{R}_{\mathrm{M}},\mathbb{R}_{\mathrm{M}},\mathbb{R}_{\mathrm{M}}:]$ is defined as follows:

(Def. 12) $[:\mathbb{R}_{\mathrm{M}},\mathbb{R}_{\mathrm{M}},\mathbb{R}_{\mathrm{M}}:] = \langle [:\mathbb{R},\mathbb{R},\mathbb{R}:], \rho^{[:\mathbb{R},\mathbb{R},\mathbb{R}:]}\rangle$.

The function $\rho^{\mathbb{R}^3}$ from $[:[:\mathbb{R}, \mathbb{R}, \mathbb{R}:], [:\mathbb{R}, \mathbb{R}, \mathbb{R}:]:]$ into $\mathbb{R}$ is defined by the condition (Def. 13).

(Def. 13) Let $x_1, y_1, x_2, y_2, x_3, y_3$ be elements of $\mathbb{R}$ and x, y be elements of $[:\mathbb{R}, \mathbb{R}, \mathbb{R}:]$. If $x = \langle x_1, x_2, x_3\rangle$ and $y = \langle y_1, y_2, y_3\rangle$, then $\rho^{\mathbb{R}^3}(x, y) = \sqrt{\rho_{\mathbb{R}}(x_1, y_1)^2 + \rho_{\mathbb{R}}(x_2, y_2)^2 + \rho_{\mathbb{R}}(x_3, y_3)^2}$.

Next we state three propositions:

(28) Let $x_1, x_2, y_1, y_2, x_3, y_3$ be elements of $\mathbb{R}$ and x, y be elements of $[:\mathbb{R}, \mathbb{R}, \mathbb{R}:]$. If $x = \langle x_1, x_2, x_3\rangle$ and $y = \langle y_1, y_2, y_3\rangle$, then $\rho^{\mathbb{R}^3}(x, y) = 0$ iff $x = y$.

(29) For all elements x, y of $[:\mathbb{R}, \mathbb{R}, \mathbb{R}:]$ such that $x = \langle x_1, x_2, x_3\rangle$ and $y = \langle y_1, y_2, y_3\rangle$ holds $\rho^{\mathbb{R}^3}(x, y) = \rho^{\mathbb{R}^3}(y, x)$.

(30) For all elements x, y, z of $[:\mathbb{R}, \mathbb{R}, \mathbb{R}:]$ such that $x = \langle x_1, x_2, x_3\rangle$ and $y = \langle y_1, y_2, y_3\rangle$ and $z = \langle z_1, z_2, z_3\rangle$ holds $\rho^{\mathbb{R}^3}(x, z) \leq \rho^{\mathbb{R}^3}(x, y) + \rho^{\mathbb{R}^3}(y, z)$.

The strict non empty metric space the Euclidean space is defined by:

(Def. 14) The Euclidean space $= \langle [:\mathbb{R}, \mathbb{R}, \mathbb{R}:], \rho^{\mathbb{R}^3}\rangle$.

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