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# On Pseudometric Spaces[1]

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**Summary.** We introduce the equivalence classes in a pseudometric space. Next we prove that the set of the equivalence classes forms the metric space with the special metric defined in the article.

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The articles [5], [2], [8], [7], [4], [1], [3], and [6] provide the notation and terminology for this paper.

Let $M$ be a non empty metric structure and let $x$, $y$ be elements of $M$. The predicate $x \approx y$ is defined by:

(Def. 1) $\rho(x,y) = 0$.

Let $M$ be a Reflexive non empty metric structure and let $x$, $y$ be elements of $M$. Let us note that the predicate $x \approx y$ is reflexive.

Let $M$ be a symmetric non empty metric structure and let $x$, $y$ be elements of $M$. Let us note that the predicate $x \approx y$ is symmetric.

Let $M$ be a non empty metric structure and let $x$ be an element of $M$. The functor $x^{\square}$ yielding a subset of $M$ is defined as follows:

(Def. 2) $x^{\square} = \{y; y$ ranges over elements of $M$: $x \approx y\}$.

Let $M$ be a non empty metric structure. A subset of $M$ is called a $\square$-equivalence class of $M$ if:

(Def. 3) There exists an element $x$ of $M$ such that it $= x^{\square}$.

Next we state a number of propositions:

(6)[1] For every pseudo metric space $M$ and for all elements $x$, $y$, $z$ of $M$ such that $x \approx y$ and $y \approx z$ holds $x \approx z$.

(7) For every pseudo metric space $M$ and for all elements $x$, $y$ of $M$ holds $y \in x^{\square}$ iff $y \approx x$.

(8) For every pseudo metric space $M$ and for all elements $x$, $p$, $q$ of $M$ such that $p \in x^{\square}$ and $q \in x^{\square}$ holds $p \approx q$.

(9) For every pseudo metric space $M$ and for every element $x$ of $M$ holds $x \in x^{\square}$.

(10) For every pseudo metric space $M$ and for all elements $x$, $y$ of $M$ holds $x \in y^{\square}$ iff $y \in x^{\square}$.

(11) For every pseudo metric space $M$ and for all elements $p$, $x$, $y$ of $M$ such that $p \in x^{\square}$ and $x \approx y$ holds $p \in y^{\square}$.

[1]Supported by RPBP.III-24.B3.
[1] The propositions (1)–(5) have been removed.

(12) For every pseudo metric space $M$ and for all elements $x$, $y$ of $M$ such that $y \in x^{\square}$ holds $x^{\square} = y^{\square}$.

(13) For every pseudo metric space $M$ and for all elements $x$, $y$ of $M$ holds $x^{\square} = y^{\square}$ iff $x \approx y$.

(14) For every pseudo metric space $M$ and for all elements $x$, $y$ of $M$ holds $x^{\square}$ meets $y^{\square}$ iff $x \approx y$.

(16)[2] For every pseudo metric space $M$ holds every $\square$-equivalence class of $M$ is a non empty set.

Let $M$ be a pseudo metric space. One can verify that every $\square$-equivalence class of $M$ is non empty.

Next we state several propositions:

(17) For every pseudo metric space $M$ and for all elements $x$, $p$, $q$ of $M$ such that $p \in x^{\square}$ and $q \in x^{\square}$ holds $\rho(p,q) = 0$.

(18) Let $M$ be a Reflexive discernible non empty metric structure and $x$, $y$ be elements of $M$. Then $x \approx y$ if and only if $x = y$.

(19) For every non empty metric space $M$ and for all elements $x$, $y$ of $M$ holds $y \in x^{\square}$ iff $y = x$.

(20) For every non empty metric space $M$ and for every element $x$ of $M$ holds $x^{\square} = \{x\}$.

(21) Let $M$ be a non empty metric space and $V$ be a subset of $M$. Then $V$ is a $\square$-equivalence class of $M$ if and only if there exists an element $x$ of $M$ such that $V = \{x\}$.

Let $M$ be a non empty metric structure. The functor $M^{\square}$ yields a set and is defined by:

(Def. 4) $M^{\square} = \{s; s \text{ ranges over elements of } 2^{\text{the carrier of } M}\colon \bigvee_{x\colon \text{element of } M} x^{\square} = s\}$.

Let $M$ be a non empty metric structure. One can check that $M^{\square}$ is non empty.

In the sequel $V$ is a set.

We now state several propositions:

(23)[3] For every non empty metric structure $M$ holds $V \in M^{\square}$ iff there exists an element $x$ of $M$ such that $V = x^{\square}$.

(24) For every non empty metric structure $M$ and for every element $x$ of $M$ holds $x^{\square} \in M^{\square}$.

(26)[4] For every non empty metric structure $M$ holds $V \in M^{\square}$ iff $V$ is a $\square$-equivalence class of $M$.

(27) For every non empty metric space $M$ and for every element $x$ of $M$ holds $\{x\} \in M^{\square}$.

(28) For every non empty metric space $M$ holds $V \in M^{\square}$ iff there exists an element $x$ of $M$ such that $V = \{x\}$.

(29) Let $M$ be a pseudo metric space, $V$, $Q$ be elements of $M^{\square}$, and $p_1$, $p_2$, $q_1$, $q_2$ be elements of $M$. If $p_1 \in V$ and $q_1 \in Q$ and $p_2 \in V$ and $q_2 \in Q$, then $\rho(p_1,q_1) = \rho(p_2,q_2)$.

Let $M$ be a non empty metric structure, let $V$, $Q$ be elements of $M^{\square}$, and let $v$ be an element of $\mathbb{R}$. We say that the distance between $V$ and $Q$ is $v$ if and only if:

(Def. 5) For all elements $p$, $q$ of $M$ such that $p \in V$ and $q \in Q$ holds $\rho(p,q) = v$.

Next we state two propositions:

(31)[5] Let $M$ be a pseudo metric space, $V$, $Q$ be elements of $M^{\square}$, and $v$ be an element of $\mathbb{R}$. Then the distance between $V$ and $Q$ is $v$ if and only if there exist elements $p$, $q$ of $M$ such that $p \in V$ and $q \in Q$ and $\rho(p,q) = v$.

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[2] The proposition (15) has been removed.
[3] The proposition (22) has been removed.
[4] The proposition (25) has been removed.
[5] The proposition (30) has been removed.

(32) Let $M$ be a pseudo metric space, $V$, $Q$ be elements of $M^{\square}$, and $v$ be an element of $\mathbb{R}$. Then the distance between $V$ and $Q$ is $v$ if and only if the distance between $Q$ and $V$ is $v$.

Let $M$ be a non empty metric structure and let $V$, $Q$ be elements of $M^{\square}$. The functor $\rho^{\circ}(V,Q)$ yields a subset of $\mathbb{R}$ and is defined by:

(Def. 6) $\rho^{\circ}(V,Q) = \{v; v$ ranges over elements of $\mathbb{R}$: the distance between $V$ and $Q$ is $v\}$.

Next we state the proposition

(34)[6] Let $M$ be a pseudo metric space, $V$, $Q$ be elements of $M^{\square}$, and $v$ be an element of $\mathbb{R}$. Then $v \in \rho^{\circ}(V,Q)$ if and only if the distance between $V$ and $Q$ is $v$.

Let $M$ be a non empty metric structure and let $v$ be an element of $\mathbb{R}$. The functor $\rho_M^{\square\,-1}(v)$ yielding a subset of $[\!:M^{\square}, M^{\square}:\!]$ is defined by the condition (Def. 7).

(Def. 7) $\rho_M^{\square\,-1}(v) = \{W; W$ ranges over elements of $[\!:M^{\square}, M^{\square}:\!]$: $\bigvee_{V,Q\,:\,\text{element of } M^{\square}} (W = \langle V, Q\rangle \wedge$ the distance between $V$ and $Q$ is $v)\}$.

Next we state the proposition

(36)[7] Let $M$ be a pseudo metric space, $v$ be an element of $\mathbb{R}$, and $W$ be an element of $[\!:M^{\square}, M^{\square}:\!]$. Then $W \in \rho_M^{\square\,-1}(v)$ if and only if there exist elements $V$, $Q$ of $M^{\square}$ such that $W = \langle V, Q\rangle$ and the distance between $V$ and $Q$ is $v$.

Let $M$ be a non empty metric structure. The functor $\rho^{\circ}(M^{\square},M^{\square})$ yields a subset of $\mathbb{R}$ and is defined by:

(Def. 8) $\rho^{\circ}(M^{\square},M^{\square}) = \{v; v$ ranges over elements of $\mathbb{R}$: $\bigvee_{V,Q\,:\,\text{element of } M^{\square}}$ the distance between $V$ and $Q$ is $v\}$.

Next we state the proposition

(38)[8] Let $M$ be a pseudo metric space and $v$ be an element of $\mathbb{R}$. Then $v \in \rho^{\circ}(M^{\square},M^{\square})$ if and only if there exist elements $V$, $Q$ of $M^{\square}$ such that the distance between $V$ and $Q$ is $v$.

Let $M$ be a non empty metric structure. The functor $\mathrm{dom}_1\, \rho_M^{\square}$ yielding a subset of $M^{\square}$ is defined by the condition (Def. 9).

(Def. 9) $\mathrm{dom}_1\, \rho_M^{\square} = \{V; V$ ranges over elements of $M^{\square}$: $\bigvee_{Q\,:\,\text{element of } M^{\square}} \bigvee_{v\,:\,\text{element of } \mathbb{R}}$ the distance between $V$ and $Q$ is $v\}$.

Next we state the proposition

(40)[9] Let $M$ be a pseudo metric space and $V$ be an element of $M^{\square}$. Then $V \in \mathrm{dom}_1\, \rho_M^{\square}$ if and only if there exists an element $Q$ of $M^{\square}$ and there exists an element $v$ of $\mathbb{R}$ such that the distance between $V$ and $Q$ is $v$.

Let $M$ be a non empty metric structure. The functor $\mathrm{dom}_2\, \rho_M^{\square}$ yields a subset of $M^{\square}$ and is defined by the condition (Def. 10).

(Def. 10) $\mathrm{dom}_2\, \rho_M^{\square} = \{Q; Q$ ranges over elements of $M^{\square}$: $\bigvee_{V\,:\,\text{element of } M^{\square}} \bigvee_{v\,:\,\text{element of } \mathbb{R}}$ the distance between $V$ and $Q$ is $v\}$.

The following proposition is true

[6] The proposition (33) has been removed.
[7] The proposition (35) has been removed.
[8] The proposition (37) has been removed.
[9] The proposition (39) has been removed.

(42)[10] Let $M$ be a pseudo metric space and $Q$ be an element of $M^{\square}$. Then $Q \in \operatorname{dom}_2 \rho^{\square}_M$ if and only if there exists an element $V$ of $M^{\square}$ and there exists an element $v$ of $\mathbb{R}$ such that the distance between $V$ and $Q$ is $v$.

Let $M$ be a non empty metric structure. The functor $\operatorname{dom} \rho^{\square}_M$ yielding a subset of $[: M^{\square}, M^{\square} :]$ is defined by the condition (Def. 11).

(Def. 11) $\operatorname{dom} \rho^{\square}_M = \{V_1; V_1$ ranges over elements of $[: M^{\square}, M^{\square} :]$: $\bigvee_{V,Q:\text{element of } M^{\square}} \bigvee_{v:\text{element of } \mathbb{R}} (V_1 = \langle V, Q \rangle \wedge$ the distance between $V$ and $Q$ is $v)\}$.

The following proposition is true

(44)[11] Let $M$ be a pseudo metric space and $V_1$ be an element of $[: M^{\square}, M^{\square} :]$. Then $V_1 \in \operatorname{dom} \rho^{\square}_M$ if and only if there exist elements $V$, $Q$ of $M^{\square}$ and there exists an element $v$ of $\mathbb{R}$ such that $V_1 = \langle V, Q \rangle$ and the distance between $V$ and $Q$ is $v$.

Let $M$ be a non empty metric structure. The functor $\operatorname{graph} \rho^{\square}_M$ yields a subset of $[: M^{\square}, M^{\square}, \mathbb{R} :]$ and is defined by the condition (Def. 12).

(Def. 12) $\operatorname{graph} \rho^{\square}_M = \{V_2; V_2$ ranges over elements of $[: M^{\square}, M^{\square}, \mathbb{R} :]$: $\bigvee_{V,Q:\text{element of } M^{\square}} \bigvee_{v:\text{element of } \mathbb{R}} (V_2 = \langle V, Q, v \rangle \wedge$ the distance between $V$ and $Q$ is $v)\}$.

We now state several propositions:

(46)[12] Let $M$ be a pseudo metric space and $V_2$ be an element of $[: M^{\square}, M^{\square}, \mathbb{R} :]$. Then $V_2 \in \operatorname{graph} \rho^{\square}_M$ if and only if there exist elements $V$, $Q$ of $M^{\square}$ and there exists an element $v$ of $\mathbb{R}$ such that $V_2 = \langle V, Q, v \rangle$ and the distance between $V$ and $Q$ is $v$.

(47) For every pseudo metric space $M$ holds $\operatorname{dom}_1 \rho^{\square}_M = \operatorname{dom}_2 \rho^{\square}_M$.

(48) For every pseudo metric space $M$ holds $\operatorname{graph} \rho^{\square}_M \subseteq [: \operatorname{dom}_1 \rho^{\square}_M, \operatorname{dom}_2 \rho^{\square}_M, \rho^{\circ}(M^{\square}, M^{\square}) :]$.

(50)[13] Let $M$ be a pseudo metric space, $V$, $Q$ be elements of $M^{\square}$, and $v_1$, $v_2$ be elements of $\mathbb{R}$. Suppose the distance between $V$ and $Q$ is $v_1$ and the distance between $V$ and $Q$ is $v_2$. Then $v_1 = v_2$.

(52)[14] Let $M$ be a pseudo metric space and $V$, $Q$ be elements of $M^{\square}$. Then there exists an element $v$ of $\mathbb{R}$ such that the distance between $V$ and $Q$ is $v$.

Let $M$ be a pseudo metric space. The functor $\rho^{\square}_M$ yields a function from $[: M^{\square}, M^{\square} :]$ into $\mathbb{R}$ and is defined as follows:

(Def. 13) For all elements $V$, $Q$ of $M^{\square}$ and for all elements $p$, $q$ of $M$ such that $p \in V$ and $q \in Q$ holds $\rho^{\square}_M(V, Q) = \rho(p, q)$.

The following three propositions are true:

(54)[15] For every pseudo metric space $M$ and for all elements $V$, $Q$ of $M^{\square}$ holds $\rho^{\square}_M(V, Q) = 0$ iff $V = Q$.

(55) For every pseudo metric space $M$ and for all elements $V$, $Q$ of $M^{\square}$ holds $\rho^{\square}_M(V, Q) = \rho^{\square}_M(Q, V)$.

(56) For every pseudo metric space $M$ and for all elements $V$, $Q$, $W$ of $M^{\square}$ holds $\rho^{\square}_M(V, W) \leq \rho^{\square}_M(V, Q) + \rho^{\square}_M(Q, W)$.

Let $M$ be a pseudo metric space. The functor $M_{/\square}$ yielding a metric space is defined by:

(Def. 14) $M_{/\square} = \langle M^{\square}, \rho^{\square}_M \rangle$.

Let $M$ be a pseudo metric space. Note that $M_{/\square}$ is strict and non empty.

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[10] The proposition (41) has been removed.
[11] The proposition (43) has been removed.
[12] The proposition (45) has been removed.
[13] The proposition (49) has been removed.
[14] The proposition (51) has been removed.
[15] The proposition (53) has been removed.

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