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The Measurability of Extended Real Valued Functions

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Summary. In this article we prove the measurablility of some extended real valued functions which are f+g, $f-g$ and so on. Moreover, we will define the simple function which are defined on the sigma field. It will play an important role for the Lebesgue integral theory.

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The articles [20], [22], [1], [21], [17], [23], [11], [2], [18], [3], [4], [5], [10], [19], [6], [7], [8], [9], [12], [13], [14], [15], and [16] provide the notation and terminology for this paper.

1. Finite Valued Function

For simplicity, we follow the rules: X denotes a non empty set, x denotes an element of X, f, g denote partial functions from X to $\overline{\mathbb{R}}$, S denotes a σ-field of subsets of X, F denotes a function from $\mathbb{Q}$ into S, p denotes a rational number, r denotes a real number, n, m denote natural numbers, and A, B denote elements of S.

Let us consider X and let us consider f. We say that f is finite if and only if:

(Def. 1) For every x such that $x \in \operatorname{dom} f$ holds $|f(x)| < +\infty$.

We now state three propositions:

(1) $f = 1\,f$.

(2) For all f, g, A such that f is finite or g is finite holds $\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g$ and $\operatorname{dom}(f-g) = \operatorname{dom} f \cap \operatorname{dom} g$.

(3) Let given f, g, F, r, A. Suppose f is finite and g is finite and for every p holds $F(p) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(p)) \cap (A \cap \text{LE-dom}(g, \overline{\mathbb{R}}(r-p)))$. Then $A \cap \text{LE-dom}(f+g, \overline{\mathbb{R}}(r)) = \bigcup \operatorname{rng} F$.

2. Measurability of $f+g$ and $f-g$

We now state several propositions:

(4) There exists a function F from $\mathbb{N}$ into $\mathbb{Q}$ such that F is one-to-one and $\operatorname{dom} F = \mathbb{N}$ and $\operatorname{rng} F = \mathbb{Q}$.

(5) Let X, Y, Z be non empty sets and F be a function from X into Z. If $X \approx Y$, then there exists a function G from Y into Z such that $\operatorname{rng} F = \operatorname{rng} G$.

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(6) Let given S, f, g, A. Suppose f is measurable on A and g is measurable on A. Then there exists a function F from $\mathbb{Q}$ into S such that for every rational number p holds $F(p) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(p)) \cap (A \cap \text{LE-dom}(g, \overline{\mathbb{R}}(r-p)))$.

(7) Let given f, g, A. Suppose f is finite and g is finite and f is measurable on A and g is measurable on A. Then $f+g$ is measurable on A.

(9)[1] For every non empty set C and for all partial functions f_1, f_2 from C to $\overline{\mathbb{R}}$ holds $f_1 - f_2 = f_1 + -f_2$.

(10) For every real number r holds $\overline{\mathbb{R}}(-r) = -\overline{\mathbb{R}}(r)$.

(11) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $-f = (-1)\,f$.

(12) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and r be a real number. If f is finite, then $r\,f$ is finite.

(13) Let given f, g, A. Suppose f is finite and g is finite and f is measurable on A and g is measurable on A and $A \subseteq \operatorname{dom} g$. Then $f-g$ is measurable on A.

3. Definitions of Extended Real Valued Functions $\max_{+}(f)$ and $\max_{-}(f)$ and their Basic Properties

Let C be a non empty set and let f be a partial function from C to $\overline{\mathbb{R}}$. The functor $\max_{+}(f)$ yielding a partial function from C to $\overline{\mathbb{R}}$ is defined by:

(Def. 2) $\operatorname{dom}\max_{+}(f) = \operatorname{dom} f$ and for every element x of C such that $x \in \operatorname{dom}\max_{+}(f)$ holds $(\max_{+}(f))(x) = \max(f(x), 0_{\overline{\mathbb{R}}})$.

The functor $\max_{-}(f)$ yields a partial function from C to $\overline{\mathbb{R}}$ and is defined as follows:

(Def. 3) $\operatorname{dom}\max_{-}(f) = \operatorname{dom} f$ and for every element x of C such that $x \in \operatorname{dom}\max_{-}(f)$ holds $(\max_{-}(f))(x) = \max(-f(x), 0_{\overline{\mathbb{R}}})$.

One can prove the following propositions:

(14) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$, then $0_{\overline{\mathbb{R}}} \leq (\max_{+}(f))(x)$.

(15) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$, then $0_{\overline{\mathbb{R}}} \leq (\max_{-}(f))(x)$.

(16) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $\max_{-}(f) = \max_{+}(-f)$.

(17) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $0_{\overline{\mathbb{R}}} < (\max_{+}(f))(x)$, then $(\max_{-}(f))(x) = 0_{\overline{\mathbb{R}}}$.

(18) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $0_{\overline{\mathbb{R}}} < (\max_{-}(f))(x)$, then $(\max_{+}(f))(x) = 0_{\overline{\mathbb{R}}}$.

(19) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $\operatorname{dom} f = \operatorname{dom}(\max_{+}(f) - \max_{-}(f))$ and $\operatorname{dom} f = \operatorname{dom}(\max_{+}(f) + \max_{-}(f))$.

(20) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$, then $(\max_{+}(f))(x) = f(x)$ or $(\max_{+}(f))(x) = 0_{\overline{\mathbb{R}}}$ but $(\max_{-}(f))(x) = -f(x)$ or $(\max_{-}(f))(x) = 0_{\overline{\mathbb{R}}}$.

(21) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $(\max_{+}(f))(x) = f(x)$, then $(\max_{-}(f))(x) = 0_{\overline{\mathbb{R}}}$.

[1] The proposition (8) has been removed.

(22) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $(\max_+(f))(x) = 0_{\overline{\mathbb{R}}}$, then $(\max_-(f))(x) = -f(x)$.

(23) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $(\max_-(f))(x) = -f(x)$, then $(\max_+(f))(x) = 0_{\overline{\mathbb{R}}}$.

(24) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and x be an element of C. If $x \in \operatorname{dom} f$ and $(\max_-(f))(x) = 0_{\overline{\mathbb{R}}}$, then $(\max_+(f))(x) = f(x)$.

(25) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $f = \max_+(f) - \max_-(f)$.

(26) For every non empty set C and for every partial function f from C to $\overline{\mathbb{R}}$ holds $|f| = \max_+(f) + \max_-(f)$.

4. Measurability of $\max_+(f)$, $\max_-(f)$ and $|f|$

Next we state three propositions:

(27) If f is measurable on A, then $\max_+(f)$ is measurable on A.

(28) If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $\max_-(f)$ is measurable on A.

(29) For all f, A such that f is measurable on A and $A \subseteq \operatorname{dom} f$ holds $|f|$ is measurable on A.

5. Definition and Measurability of Characteristic Function

Next we state the proposition

(30) For all sets A, X holds $\operatorname{rng}(\chi_{A,X}) \subseteq \{0_{\overline{\mathbb{R}}}, \overline{1}\}$.

Let A, X be sets. Then $\chi_{A,X}$ is a partial function from X to $\overline{\mathbb{R}}$.
One can prove the following two propositions:

(31) $\chi_{A,X}$ is finite.

(32) $\chi_{A,X}$ is measurable on B.

6. Definition and Measurability of Simple Function

Let X be a set and let S be a σ-field of subsets of X. Note that there exists a finite sequence of elements of S which is disjoint valued.

Let X be a set and let S be a σ-field of subsets of X. A finite sequence of separated subsets of S is a disjoint valued finite sequence of elements of S.

We now state two propositions:

(33) Suppose F is a finite sequence of separated subsets of S. Then there exists a sequence G of separated subsets of S such that $\bigcup \operatorname{rng} F = \bigcup \operatorname{rng} G$ and for every n such that $n \in \operatorname{dom} F$ holds $F(n) = G(n)$ and for every m such that $m \notin \operatorname{dom} F$ holds $G(m) = \emptyset$.

(34) If F is a finite sequence of separated subsets of S, then $\bigcup \operatorname{rng} F \in S$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let f be a partial function from X to $\overline{\mathbb{R}}$. We say that f is simple function in S if and only if the conditions (Def. 5) are satisfied.

(Def. 5)[2] (i) f is finite, and

(ii) there exists a finite sequence F of separated subsets of S such that $\operatorname{dom} f = \bigcup \operatorname{rng} F$ and for every natural number n and for all elements x, y of X such that $n \in \operatorname{dom} F$ and $x \in F(n)$ and $y \in F(n)$ holds $f(x) = f(y)$.

[2] The definition (Def. 4) has been removed.

One can prove the following propositions:

(35) If f is finite, then $\mathrm{rng}\, f$ is a subset of $\mathbb{R}$.

(36) Suppose F is a finite sequence of separated subsets of S. Let given n. Then $F \restriction \mathrm{Seg}\, n$ is a finite sequence of separated subsets of S.

(37) If f is simple function in S, then f is measurable on A.

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