JOURNAL OF FORMALIZED MATHEMATICS
Volume 5, Released 1993, Published 2003
Inst. of Computer Science, Univ. of Białystok

Properties of the Intervals of Real Numbers

Józef Białas
Łódź University

Summary. The paper contains definitions and basic properties of the intervals of real numbers.

The article includes the text being a continuation of the paper [4]. Some theorems concerning basic properties of intervals are proved.

MML Identifier: MEASURE5.

WWW: http://mizar.org/JFM/Vol5/measure5.html

The articles [5], [6], [1], [2], and [3] provide the notation and terminology for this paper.

In this paper x, y, a, b, a_1, b_1, a_2, b_2 are extended real numbers.

The following four propositions are true:

(1) If $x \neq -\infty$ and $x \neq +\infty$ and $x \leq y$, then $0_{\overline{\mathbb{R}}} \leq y - x$.

(2) If $x = -\infty$ and $y = -\infty$ and $x = +\infty$ and $y = +\infty$ and $x \leq y$, then $0_{\overline{\mathbb{R}}} \leq y - x$.

(8)[1] For all extended real numbers a, b, c such that $b \neq -\infty$ and $b \neq +\infty$ and $a = -\infty$ and $c = -\infty$ and $a = +\infty$ and $c = +\infty$ holds $(c - b) + (b - a) = c - a$.

(9) $\inf\{a_1, a_2\} \leq a_1$ and $\inf\{a_1, a_2\} \leq a_2$ and $a_1 \leq \sup\{a_1, a_2\}$ and $a_2 \leq \sup\{a_1, a_2\}$.

Let a, b be extended real numbers. The functor $[a, b]$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def. 1) For every extended real number x holds $x \in [a, b]$ iff $a \leq x$ and $x \leq b$ and $x \in \mathbb{R}$.

The functor $]a, b[$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def. 2) For every extended real number x holds $x \in]a, b[$ iff $a < x$ and $x < b$ and $x \in \mathbb{R}$.

The functor $]a, b]$ yielding a subset of $\mathbb{R}$ is defined by:

(Def. 3) For every extended real number x holds $x \in]a, b]$ iff $a < x$ and $x \leq b$ and $x \in \mathbb{R}$.

The functor $[a, b[$ yielding a subset of $\mathbb{R}$ is defined by:

(Def. 4) For every extended real number x holds $x \in [a, b[$ iff $a \leq x$ and $x < b$ and $x \in \mathbb{R}$.

Let I_1 be a subset of $\mathbb{R}$. We say that I_1 is open interval if and only if:

(Def. 5) There exist extended real numbers a, b such that $a \leq b$ and $I_1 =]a, b[$.

We say that I_1 is closed interval if and only if:

[1] The propositions (3)–(7) have been removed.

(Def. 6) There exist extended real numbers a, b such that $a \leq b$ and $I_1 = [a,b]$.

Let us mention that there exists a subset of $\mathbb{R}$ which is open interval and there exists a subset of $\mathbb{R}$ which is closed interval.

Let I_1 be a subset of $\mathbb{R}$. We say that I_1 is right open interval if and only if:

(Def. 7) There exist extended real numbers a, b such that $a \leq b$ and $I_1 = [a,b[$.

We introduce I_1 is left closed interval as a synonym of I_1 is right open interval.

Let I_1 be a subset of $\mathbb{R}$. We say that I_1 is left open interval if and only if:

(Def. 8) There exist extended real numbers a, b such that $a \leq b$ and $I_1 =]a,b]$.

We introduce I_1 is right closed interval as a synonym of I_1 is left open interval.

Let us observe that there exists a subset of $\mathbb{R}$ which is right open interval and there exists a subset of $\mathbb{R}$ which is left open interval.

Let I_1 be a subset of $\mathbb{R}$. We say that I_1 is interval if and only if:

(Def. 9) I_1 is open interval, closed interval, right open interval, and left open interval.

One can check that there exists a subset of $\mathbb{R}$ which is interval.

An interval is an interval subset of $\mathbb{R}$.

In the sequel A, B are intervals.

One can verify the following observations:

* every subset of $\mathbb{R}$ which is open interval is also interval,
* every subset of $\mathbb{R}$ which is closed interval is also interval,
* every subset of $\mathbb{R}$ which is right open interval is also interval, and
* every subset of $\mathbb{R}$ which is left open interval is also interval.

We now state a number of propositions:

(11)[2] Let x be a set and a, b be extended real numbers. Suppose $x \in]a,b[$ or $x \in [a,b]$ or $x \in [a,b[$ or $x \in]a,b]$. Then x is an extended real number.

(12) For all extended real numbers a, b such that $b < a$ holds $]a,b[= \emptyset$ and $[a,b] = \emptyset$ and $[a,b[= \emptyset$ and $]a,b] = \emptyset$.

(13) For every extended real number a holds $]a,a[= \emptyset$ and $[a,a[= \emptyset$ and $]a,a] = \emptyset$.

(14) For every extended real number a holds if $a = -\infty$ or $a = +\infty$, then $[a,a] = \emptyset$ and if $a \neq -\infty$ and $a \neq +\infty$, then $[a,a] = \{a\}$.

(15) For all extended real numbers a, b such that $b \leq a$ holds $]a,b[= \emptyset$ and $[a,b[= \emptyset$ and $]a,b] = \emptyset$ and $[a,b] \subseteq \{a\}$ and $[a,b] \subseteq \{b\}$.

(16) For all extended real numbers a, b, c such that $a < b$ and $b < c$ holds $b \in \mathbb{R}$.

(17) Let a, b be extended real numbers. Suppose $a < b$. Then there exists an extended real number x such that $a < x$ and $x < b$ and $x \in \mathbb{R}$.

(18) Let a, b, c be extended real numbers. Suppose $a < b$ and $a < c$. Then there exists an extended real number x such that $a < x$ and $x < b$ and $x < c$ and $x \in \mathbb{R}$.

(19) Let a, b, c be extended real numbers. Suppose $a < c$ and $b < c$. Then there exists an extended real number x such that $a < x$ and $b < x$ and $x < c$ and $x \in \mathbb{R}$.

(20) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1[$ and $x \notin]a_2,b_2[$ or $x \notin]a_1,b_1[$ and $x \in]a_2,b_2[$.

[2] The proposition (10) has been removed.

(21) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1[$ and $x \notin]a_2,b_2[$ or $x \notin]a_1,b_1[$ and $x \in]a_2,b_2[$.

(22) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1]$ and $x \notin]a_2,b_2[$ or $x \notin [a_1,b_1]$ and $x \in]a_2,b_2[$.

(23) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1]$ and $x \notin]a_2,b_2[$ or $x \notin [a_1,b_1]$ and $x \in]a_2,b_2[$.

(24) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1[$ and $x \notin [a_2,b_2]$ or $x \notin]a_1,b_1[$ and $x \in [a_2,b_2]$.

(25) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1[$ and $x \notin [a_2,b_2]$ or $x \notin]a_1,b_1[$ and $x \in [a_2,b_2]$.

(26) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1[$ and $x \notin [a_2,b_2[$ or $x \notin]a_1,b_1[$ and $x \in [a_2,b_2[$.

(27) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1[$ and $x \notin [a_2,b_2[$ or $x \notin]a_1,b_1[$ and $x \in [a_2,b_2[$.

(28) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1[$ and $x \notin]a_2,b_2[$ or $x \notin [a_1,b_1[$ and $x \in]a_2,b_2[$.

(29) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1[$ and $x \notin]a_2,b_2[$ or $x \notin [a_1,b_1[$ and $x \in]a_2,b_2[$.

(30) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1[$ and $x \notin]a_2,b_2]$ or $x \notin]a_1,b_1[$ and $x \in]a_2,b_2]$.

(31) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1[$ and $x \notin]a_2,b_2]$ or $x \notin]a_1,b_1[$ and $x \in]a_2,b_2]$.

(32) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1]$ and $x \notin]a_2,b_2[$ or $x \notin]a_1,b_1]$ and $x \in]a_2,b_2[$.

(33) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1]$ and $x \notin]a_2,b_2[$ or $x \notin]a_1,b_1]$ and $x \in]a_2,b_2[$.

(34) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1]$ and $x \notin [a_2,b_2]$ or $x \notin [a_1,b_1]$ and $x \in [a_2,b_2]$.

(35) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1]$ and $x \notin [a_2,b_2]$ or $x \notin [a_1,b_1]$ and $x \in [a_2,b_2]$.

(36) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1]$ and $x \notin [a_2,b_2[$ or $x \notin [a_1,b_1]$ and $x \in [a_2,b_2[$.

(37) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1]$ and $x \notin [a_2,b_2[$ or $x \notin [a_1,b_1]$ and $x \in [a_2,b_2[$.

(38) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1[$ and $x \notin [a_2,b_2]$ or $x \notin [a_1,b_1[$ and $x \in [a_2,b_2]$.

(39) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1[$ and $x \notin [a_2,b_2]$ or $x \notin [a_1,b_1[$ and $x \in [a_2,b_2]$.

(40) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1]$ and $x \notin]a_2,b_2]$ or $x \notin [a_1,b_1]$ and $x \in]a_2,b_2]$.

(41) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1]$ and $x \notin]a_2,b_2]$ or $x \notin [a_1,b_1]$ and $x \in]a_2,b_2]$.

(42) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1]$ and $x \notin [a_2,b_2]$ or $x \notin]a_1,b_1]$ and $x \in [a_2,b_2]$.

(43) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1]$ and $x \notin [a_2,b_2]$ or $x \notin]a_1,b_1]$ and $x \in [a_2,b_2]$.

(44) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1[$ and $x \notin [a_2,b_2[$ or $x \notin [a_1,b_1[$ and $x \in [a_2,b_2[$.

(45) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1[$ and $x \notin [a_2,b_2[$ or $x \notin [a_1,b_1[$ and $x \in [a_2,b_2[$.

(46) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1[$ and $x \notin]a_2,b_2]$ or $x \notin [a_1,b_1[$ and $x \in]a_2,b_2]$.

(47) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in [a_1,b_1[$ and $x \notin]a_2,b_2]$ or $x \notin [a_1,b_1[$ and $x \in]a_2,b_2]$.

(48) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1]$ and $x \notin [a_2,b_2[$ or $x \notin]a_1,b_1]$ and $x \in [a_2,b_2[$.

(49) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1]$ and $x \notin [a_2,b_2[$ or $x \notin]a_1,b_1]$ and $x \in [a_2,b_2[$.

(50) If $a_1 < a_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1]$ and $x \notin]a_2,b_2]$ or $x \notin]a_1,b_1]$ and $x \in]a_2,b_2]$.

(51) If $b_1 < b_2$ and if $a_1 < b_1$ or $a_2 < b_2$, then there exists an extended real number x such that $x \in]a_1,b_1]$ and $x \notin]a_2,b_2]$ or $x \notin]a_1,b_1]$ and $x \in]a_2,b_2]$.

(52) If $a_1 < b_1$ and if $A =]a_1,b_1[$ or $A = [a_1,b_1]$ or $A = [a_1,b_1[$ or $A =]a_1,b_1]$ and if $A =]a_2,b_2[$ or $A = [a_2,b_2]$ or $A = [a_2,b_2[$ or $A =]a_2,b_2]$, then $a_1 = a_2$ and $b_1 = b_2$.

Let A be an interval. The functor $\mathrm{vol}(A)$ yields an extended real number and is defined by the condition (Def. 10).

(Def. 10) There exist extended real numbers a, b such that $A =]a,b[$ or $A = [a,b]$ or $A = [a,b[$ or $A =]a,b]$ but if $a < b$, then $\mathrm{vol}(A) = b - a$ but if $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

We now state several propositions:

(53) Let A be an open interval subset of $\mathbb{R}$ and a, b be extended real numbers such that $A =]a,b[$. Then

(i) if $a < b$, then $\mathrm{vol}(A) = b - a$, and

(ii) if $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

(54) Let A be a closed interval subset of $\mathbb{R}$ and a, b be extended real numbers such that $A = [a,b]$. Then

(i) if $a < b$, then $\mathrm{vol}(A) = b - a$, and

(ii) if $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

(55) Let A be a right open interval subset of $\mathbb{R}$ and a, b be extended real numbers such that $A = [a,b[$. Then

(i) if $a < b$, then $\mathrm{vol}(A) = b - a$, and

(ii) if $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

(56) Let A be a left open interval subset of $\mathbb{R}$ and a, b be extended real numbers such that $A =]a,b]$. Then

(i) if $a < b$, then $\mathrm{vol}(A) = b - a$, and

(ii) if $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

(57) Let a, b, c be extended real numbers. Suppose $a = -\infty$ and $b \in \mathbb{R}$ and $c = +\infty$ and $A =]a,b[$ or $A =]b,c[$ or $A = [a,b]$ or $A = [b,c]$ or $A = [a,b[$ or $A = [b,c[$ or $A =]a,b]$ or $A =]b,c]$. Then $\mathrm{vol}(A) = +\infty$.

(58) For all extended real numbers a, b such that $a = -\infty$ but $b = +\infty$ but $A =]a,b[$ or $A = [a,b]$ or $A = [a,b[$ or $A =]a,b]$ holds $\mathrm{vol}(A) = +\infty$.

One can verify that there exists an interval which is empty.
$\emptyset$ is an empty interval.
One can prove the following four propositions:

(60)[3] $\mathrm{vol}(\emptyset) = 0_{\overline{\mathbb{R}}}$.

(61) If $A \subseteq B$ and $B = [a,b]$ and $b \leq a$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$ and $\mathrm{vol}(B) = 0_{\overline{\mathbb{R}}}$.

(62) If $A \subseteq B$, then $\mathrm{vol}(A) \leq \mathrm{vol}(B)$.

(63) $0_{\overline{\mathbb{R}}} \leq \mathrm{vol}(A)$.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[2] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/supinf_1.html`.

[3] Józef Białas. Series of positive real numbers. Measure theory. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/supinf_2.html`.

[4] Józef Białas. Properties of Caratheodor's measure. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/measure4.html`.

[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[6] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

Received January 12, 1993

Published January 2, 2004

[3] The proposition (59) has been removed.