JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

N-Tuples and Cartesian Products for $n = 6$[1]

Michał Muzalewski
Warsaw University
Białystok

Wojciech Skaba
Nicolaus Copernicus University
Toruń

Summary. This article defines ordered n-tuples, projections and Cartesian products for $n = 6$. We prove many theorems concerning the basic properties of the n-tuples and Cartesian products that may be utilized in several further, more challenging applications. A few of these theorems are a strightforward consequence of the regularity axiom. The article originated as an upgrade of the article [4].

MML Identifier: MCART_3.

WWW: http://mizar.org/JFM/Vol2/mcart_3.html

The articles [3], [1], [5], [4], and [2] provide the notation and terminology for this paper.

For simplicity, we follow the rules: v, z, x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4, y_5, y_6, X, X_1, X_2, X_3, X_4, X_5, X_6, Y, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Z denote sets, x_7 denotes an element of X_1, x_8 denotes an element of X_2, x_9 denotes an element of X_3, x_{10} denotes an element of X_4, and x_{11} denotes an element of X_5.

Next we state two propositions:

(1) Suppose $X \neq \emptyset$. Then there exists Y such that

(i) $Y \in X$, and

(ii) for all Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8 such that $Y_1 \in Y_2$ and $Y_2 \in Y_3$ and $Y_3 \in Y_4$ and $Y_4 \in Y_5$ and $Y_5 \in Y_6$ and $Y_6 \in Y_7$ and $Y_7 \in Y_8$ and $Y_8 \in Y$ holds Y_1 misses X.

(2) Suppose $X \neq \emptyset$. Then there exists Y such that

(i) $Y \in X$, and

(ii) for all Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9 such that $Y_1 \in Y_2$ and $Y_2 \in Y_3$ and $Y_3 \in Y_4$ and $Y_4 \in Y_5$ and $Y_5 \in Y_6$ and $Y_6 \in Y_7$ and $Y_7 \in Y_8$ and $Y_8 \in Y_9$ and $Y_9 \in Y$ holds Y_1 misses X.

Let us consider x_1, x_2, x_3, x_4, x_5, x_6. The functor $\langle x_1,x_2,x_3,x_4,x_5,x_6\rangle$ is defined as follows:

(Def. 1) $\langle x_1,x_2,x_3,x_4,x_5,x_6\rangle = \langle\langle x_1,x_2,x_3,x_4,x_5\rangle, x_6\rangle$.

We now state several propositions:

(3) $\langle x_1,x_2,x_3,x_4,x_5,x_6\rangle = \langle\langle\langle\langle\langle x_1, x_2\rangle, x_3\rangle, x_4\rangle, x_5\rangle, x_6\rangle$.

(5)[1] $\langle x_1,x_2,x_3,x_4,x_5,x_6\rangle = \langle\langle x_1,x_2,x_3,x_4\rangle, x_5, x_6\rangle$.

(6) $\langle x_1,x_2,x_3,x_4,x_5,x_6\rangle = \langle\langle x_1, x_2, x_3\rangle,x_4,x_5,x_6\rangle$.

(7) $\langle x_1,x_2,x_3,x_4,x_5,x_6\rangle = \langle\langle x_1, x_2\rangle,x_3,x_4,x_5,x_6\rangle$.

[1] Supported by RPBP.III-24.C6.
[1] The proposition (4) has been removed.

(8) If $\langle x_1, x_2, x_3, x_4, x_5, x_6\rangle = \langle y_1, y_2, y_3, y_4, y_5, y_6\rangle$, then $x_1 = y_1$ and $x_2 = y_2$ and $x_3 = y_3$ and $x_4 = y_4$ and $x_5 = y_5$ and $x_6 = y_6$.

(9) If $X \neq \emptyset$, then there exists v such that $v \in X$ and it is not true that there exist x_1, x_2, x_3, x_4, x_5, x_6 such that $x_1 \in X$ or $x_2 \in X$ but $v = \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle$.

Let us consider X_1, X_2, X_3, X_4, X_5, X_6. The functor $[: X_1, X_2, X_3, X_4, X_5\ X_6 :]$ yields a set and is defined as follows:

(Def. 2) $[: X_1, X_2, X_3, X_4, X_5\ X_6 :] = [: [: X_1, X_2, X_3, X_4, X_5 :], X_6 :]$.

One can prove the following propositions:

(10) $[: X_1, X_2, X_3, X_4, X_5\ X_6 :] = [: [: [: [: [: X_1, X_2 :], X_3 :], X_4 :], X_5 :], X_6 :]$.

(12)[2] $[: X_1, X_2, X_3, X_4, X_5\ X_6 :] = [: [: X_1, X_2, X_3, X_4 :], X_5, X_6 :]$.

(13) $[: X_1, X_2, X_3, X_4, X_5\ X_6 :] = [: [: X_1, X_2, X_3 :], X_4, X_5, X_6 :]$.

(14) $[: X_1, X_2, X_3, X_4, X_5\ X_6 :] = [: [: X_1, X_2 :], X_3, X_4, X_5, X_6 :]$.

(15) $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $X_6 \neq \emptyset$ iff $[: X_1, X_2, X_3, X_4, X_5\ X_6 :] \neq \emptyset$.

(16) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $X_6 \neq \emptyset$. If $[: X_1, X_2, X_3, X_4, X_5\ X_6 :] = [: Y_1, Y_2, Y_3, Y_4, Y_5\ Y_6 :]$, then $X_1 = Y_1$ and $X_2 = Y_2$ and $X_3 = Y_3$ and $X_4 = Y_4$ and $X_5 = Y_5$ and $X_6 = Y_6$.

(17) If $[: X_1, X_2, X_3, X_4, X_5\ X_6 :] \neq \emptyset$ and $[: X_1, X_2, X_3, X_4, X_5\ X_6 :] = [: Y_1, Y_2, Y_3, Y_4, Y_5\ Y_6 :]$, then $X_1 = Y_1$ and $X_2 = Y_2$ and $X_3 = Y_3$ and $X_4 = Y_4$ and $X_5 = Y_5$ and $X_6 = Y_6$.

(18) If $[: X, X, X, X, X\ X :] = [: Y, Y, Y, Y, Y\ Y :]$, then $X = Y$.

In the sequel x_{12} denotes an element of X_6.

We now state the proposition

(19) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $X_6 \neq \emptyset$. Let x be an element of $[: X_1, X_2, X_3, X_4, X_5\ X_6 :]$. Then there exist x_7, x_8, x_9, x_{10}, x_{11}, x_{12} such that $x = \langle x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\rangle$.

Let us consider X_1, X_2, X_3, X_4, X_5, X_6. Let us assume that $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $X_6 \neq \emptyset$. Let x be an element of $[: X_1, X_2, X_3, X_4, X_5\ X_6 :]$. The functor $x_{\mathbf{1}}$ yielding an element of X_1 is defined as follows:

(Def. 3) If $x = \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle$, then $x_{\mathbf{1}} = x_1$.

The functor $x_{\mathbf{2}}$ yields an element of X_2 and is defined as follows:

(Def. 4) If $x = \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle$, then $x_{\mathbf{2}} = x_2$.

The functor $x_{\mathbf{3}}$ yielding an element of X_3 is defined as follows:

(Def. 5) If $x = \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle$, then $x_{\mathbf{3}} = x_3$.

The functor $x_{\mathbf{4}}$ yielding an element of X_4 is defined as follows:

(Def. 6) If $x = \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle$, then $x_{\mathbf{4}} = x_4$.

The functor $x_{\mathbf{5}}$ yielding an element of X_5 is defined as follows:

(Def. 7) If $x = \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle$, then $x_{\mathbf{5}} = x_5$.

The functor $x_{\mathbf{6}}$ yielding an element of X_6 is defined by:

[2] The proposition (11) has been removed.

(Def. 8) If $x = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$, then $x_{\mathbf{6}} = x_6$.

We now state several propositions:

(20) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $X_6 \neq \emptyset$. Let x be an element of $[: X_1, X_2, X_3, X_4, X_5\ X_6 :]$ and given x_1, x_2, x_3, x_4, x_5, x_6. If $x = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$, then $x_{\mathbf{1}} = x_1$ and $x_{\mathbf{2}} = x_2$ and $x_{\mathbf{3}} = x_3$ and $x_{\mathbf{4}} = x_4$ and $x_{\mathbf{5}} = x_5$ and $x_{\mathbf{6}} = x_6$.

(21) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $X_6 \neq \emptyset$, then for every element x of $[: X_1, X_2, X_3, X_4, X_5\ X_6 :]$ holds $x = \langle x_{\mathbf{1}}, x_{\mathbf{2}}, x_{\mathbf{3}}, x_{\mathbf{4}}, x_{\mathbf{5}}, x_{\mathbf{6}} \rangle$.

(22) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $X_6 \neq \emptyset$. Let x be an element of $[: X_1, X_2, X_3, X_4, X_5\ X_6 :]$. Then

(i) $x_{\mathbf{1}} = (((((x\ \mathbf{qua}\ \mathrm{set})_{\mathbf{1}})_{\mathbf{1}})_{\mathbf{1}})_{\mathbf{1}})_{\mathbf{1}}$,

(ii) $x_{\mathbf{2}} = (((((x\ \mathbf{qua}\ \mathrm{set})_{\mathbf{1}})_{\mathbf{1}})_{\mathbf{1}})_{\mathbf{1}})_{\mathbf{2}}$,

(iii) $x_{\mathbf{3}} = ((((x\ \mathbf{qua}\ \mathrm{set})_{\mathbf{1}})_{\mathbf{1}})_{\mathbf{1}})_{\mathbf{2}}$,

(iv) $x_{\mathbf{4}} = (((x\ \mathbf{qua}\ \mathrm{set})_{\mathbf{1}})_{\mathbf{1}})_{\mathbf{2}}$,

(v) $x_{\mathbf{5}} = ((x\ \mathbf{qua}\ \mathrm{set})_{\mathbf{1}})_{\mathbf{2}}$, and

(vi) $x_{\mathbf{6}} = (x\ \mathbf{qua}\ \mathrm{set})_{\mathbf{2}}$.

(23) If $X_1 \subseteq [: X_1, X_2, X_3, X_4, X_5\ X_6 :]$ or $X_1 \subseteq [: X_2, X_3, X_4, X_5, X_6\ X_1 :]$ or $X_1 \subseteq [: X_3, X_4, X_5, X_6, X_1\ X_2 :]$ or $X_1 \subseteq [: X_4, X_5, X_6, X_1, X_2\ X_3 :]$ or $X_1 \subseteq [: X_5, X_6, X_1, X_2, X_3\ X_4 :]$ or $X_1 \subseteq [: X_6, X_1, X_2, X_3, X_4\ X_5 :]$, then $X_1 = \emptyset$.

(24) Suppose $[: X_1, X_2, X_3, X_4, X_5\ X_6 :]$ meets $[: Y_1, Y_2, Y_3, Y_4, Y_5\ Y_6 :]$. Then X_1 meets Y_1 and X_2 meets Y_2 and X_3 meets Y_3 and X_4 meets Y_4 and X_5 meets Y_5 and X_6 meets Y_6.

(25) $[: \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}\ \{x_6\} :] = \{\langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle\}$.

For simplicity, we use the following convention: A_1 is a subset of X_1, A_2 is a subset of X_2, A_3 is a subset of X_3, A_4 is a subset of X_4, A_5 is a subset of X_5, A_6 is a subset of X_6, and x is an element of $[: X_1, X_2, X_3, X_4, X_5\ X_6 :]$.

The following propositions are true:

(26) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $X_6 \neq \emptyset$. Let given x_1, x_2, x_3, x_4, x_5, x_6. If $x = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$, then $x_{\mathbf{1}} = x_1$ and $x_{\mathbf{2}} = x_2$ and $x_{\mathbf{3}} = x_3$ and $x_{\mathbf{4}} = x_4$ and $x_{\mathbf{5}} = x_5$ and $x_{\mathbf{6}} = x_6$.

(27) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $X_6 \neq \emptyset$ and for all x_7, x_8, x_9, x_{10}, x_{11}, x_{12} such that $x = \langle x_7, x_8, x_9, x_{10}, x_{11}, x_{12} \rangle$ holds $y_1 = x_7$, then $y_1 = x_{\mathbf{1}}$.

(28) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $X_6 \neq \emptyset$ and for all x_7, x_8, x_9, x_{10}, x_{11}, x_{12} such that $x = \langle x_7, x_8, x_9, x_{10}, x_{11}, x_{12} \rangle$ holds $y_2 = x_8$, then $y_2 = x_{\mathbf{2}}$.

(29) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $X_6 \neq \emptyset$ and for all x_7, x_8, x_9, x_{10}, x_{11}, x_{12} such that $x = \langle x_7, x_8, x_9, x_{10}, x_{11}, x_{12} \rangle$ holds $y_3 = x_9$, then $y_3 = x_{\mathbf{3}}$.

(30) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $X_6 \neq \emptyset$ and for all x_7, x_8, x_9, x_{10}, x_{11}, x_{12} such that $x = \langle x_7, x_8, x_9, x_{10}, x_{11}, x_{12} \rangle$ holds $y_4 = x_{10}$, then $y_4 = x_{\mathbf{4}}$.

(31) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $X_6 \neq \emptyset$ and for all x_7, x_8, x_9, x_{10}, x_{11}, x_{12} such that $x = \langle x_7, x_8, x_9, x_{10}, x_{11}, x_{12} \rangle$ holds $y_5 = x_{11}$, then $y_5 = x_{\mathbf{5}}$.

(32) If $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $X_6 \neq \emptyset$ and for all x_7, x_8, x_9, x_{10}, x_{11}, x_{12} such that $x = \langle x_7, x_8, x_9, x_{10}, x_{11}, x_{12} \rangle$ holds $y_6 = x_{12}$, then $y_6 = x_{\mathbf{6}}$.

(33) If $z \in [: X_1, X_2, X_3, X_4, X_5\ X_6 :]$, then there exist x_1, x_2, x_3, x_4, x_5, x_6 such that $x_1 \in X_1$ and $x_2 \in X_2$ and $x_3 \in X_3$ and $x_4 \in X_4$ and $x_5 \in X_5$ and $x_6 \in X_6$ and $z = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$.

(34) $\langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle \in [: X_1, X_2, X_3, X_4, X_5\ X_6 :]$ iff $x_1 \in X_1$ and $x_2 \in X_2$ and $x_3 \in X_3$ and $x_4 \in X_4$ and $x_5 \in X_5$ and $x_6 \in X_6$.

(35) Suppose that for every z holds $z \in Z$ iff there exist $x_1, x_2, x_3, x_4, x_5, x_6$ such that $x_1 \in X_1$ and $x_2 \in X_2$ and $x_3 \in X_3$ and $x_4 \in X_4$ and $x_5 \in X_5$ and $x_6 \in X_6$ and $z = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$. Then $Z = [: X_1, X_2, X_3, X_4, X_5\ X_6 :]$.

(36) Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$ and $X_4 \neq \emptyset$ and $X_5 \neq \emptyset$ and $X_6 \neq \emptyset$ and $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$ and $Y_3 \neq \emptyset$ and $Y_4 \neq \emptyset$ and $Y_5 \neq \emptyset$ and $Y_6 \neq \emptyset$. Let x be an element of $[: X_1, X_2, X_3, X_4, X_5\ X_6 :]$ and y be an element of $[: Y_1, Y_2, Y_3, Y_4, Y_5\ Y_6 :]$. If $x = y$, then $x_{\mathbf{1}} = y_{\mathbf{1}}$ and $x_{\mathbf{2}} = y_{\mathbf{2}}$ and $x_{\mathbf{3}} = y_{\mathbf{3}}$ and $x_{\mathbf{4}} = y_{\mathbf{4}}$ and $x_{\mathbf{5}} = y_{\mathbf{5}}$ and $x_{\mathbf{6}} = y_{\mathbf{6}}$.

(37) For every element x of $[: X_1, X_2, X_3, X_4, X_5\ X_6 :]$ such that $x \in [: A_1, A_2, A_3, A_4, A_5\ A_6 :]$ holds $x_{\mathbf{1}} \in A_1$ and $x_{\mathbf{2}} \in A_2$ and $x_{\mathbf{3}} \in A_3$ and $x_{\mathbf{4}} \in A_4$ and $x_{\mathbf{5}} \in A_5$ and $x_{\mathbf{6}} \in A_6$.

(38) If $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$ and $X_3 \subseteq Y_3$ and $X_4 \subseteq Y_4$ and $X_5 \subseteq Y_5$ and $X_6 \subseteq Y_6$, then $[: X_1, X_2, X_3, X_4, X_5\ X_6 :] \subseteq [: Y_1, Y_2, Y_3, Y_4, Y_5\ Y_6 :]$.

(39) $[: A_1, A_2, A_3, A_4, A_5\ A_6 :]$ is a subset of $[: X_1, X_2, X_3, X_4, X_5\ X_6 :]$.

REFERENCES

[1] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[2] Michał Muzalewski and Wojciech Skaba. n-tuples and Cartesian products for $n = 5$. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/mcart_2.html.

[3] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[4] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/mcart_1.html.

[5] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

Received October 15, 1990

Published January 2, 2004