JOURNAL OF FORMALIZED MATHEMATICS
Volume 5, Released 1993, Published 2003
Inst. of Computer Science, Univ. of Białystok

The Product and the Determinant of Matrices with Entries in a Field

Katarzyna Zawadzka
Warsaw University
Białystok

Summary. Concerned with a generalization of concepts introduced in [14], i.e. there are introduced the sum and the product of matrices of any dimension of elements of any field.

MML Identifier: MATRIX_3.

WWW: http://mizar.org/JFM/Vol5/matrix_3.html

The articles [19], [8], [24], [20], [13], [25], [6], [7], [3], [5], [4], [17], [23], [16], [18], [11], [10], [9], [14], [22], [15], [1], [21], [26], [2], and [12] provide the notation and terminology for this paper.

1. AUXILIARY THEOREMS

We adopt the following rules: i, j, k, l, n, m are natural numbers, D is a non empty set, and K is a field.

The following proposition is true

(1) If $n = n + k$, then $k = 0$.

Let us consider K, n, m. The functor $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times m}$ yields a matrix over K of dimension $n \times m$ and is defined by:

(Def. 1) $$\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times m} = n \mapsto (m \mapsto 0_K).$$

Let us consider K and let A be a matrix over K. The functor $-A$ yields a matrix over K and is defined by:

(Def. 2) $\mathrm{len}(-A) = \mathrm{len}A$ and $\mathrm{width}(-A) = \mathrm{width}A$ and for all i, j such that $\langle i, j\rangle \in$ the indices of A holds $(-A) \circ (i, j) = -(A \circ (i, j))$.

Let us consider K and let A, B be matrices over K. The functor $A + B$ yields a matrix over K and is defined by:

(Def. 3) $\mathrm{len}(A+B) = \mathrm{len}A$ and $\mathrm{width}(A+B) = \mathrm{width}A$ and for all i, j such that $\langle i, j\rangle \in$ the indices of A holds $(A+B) \circ (i, j) = (A \circ (i, j)) + (B \circ (i, j))$.

The following propositions are true:

(3)[1] For all i, j such that $\langle i, j\rangle \in$ the indices of $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n\times m}$ holds $\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n\times m} \circ (i, j) = 0_K$.

(4) For all matrices A, B over K such that $\mathrm{len}A = \mathrm{len}B$ and $\mathrm{width}A = \mathrm{width}B$ holds $A + B = B + A$.

(5) For all matrices A, B, C over K such that $\mathrm{len}A = \mathrm{len}B$ and $\mathrm{len}A = \mathrm{len}C$ and $\mathrm{width}A = \mathrm{width}B$ and $\mathrm{width}A = \mathrm{width}C$ holds $(A+B)+C = A+(B+C)$.

(6) For every matrix A over K of dimension $n \times m$ holds $A + \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n\times m} = A$.

(7) For every matrix A over K of dimension $n \times m$ holds $A + -A = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{n\times m}$.

Let us consider K and let A, B be matrices over K. Let us assume that $\mathrm{width}A = \mathrm{len}B$. The functor $A \cdot B$ yielding a matrix over K is defined by:

(Def. 4) $\mathrm{len}(A \cdot B) = \mathrm{len}A$ and $\mathrm{width}(A \cdot B) = \mathrm{width}B$ and for all i, j such that $\langle i, j\rangle \in$ the indices of $A \cdot B$ holds $(A \cdot B) \circ (i, j) = \mathrm{Line}(A, i) \cdot B_{\square, j}$.

Let us consider n, k, m, let us consider K, let A be a matrix over K of dimension $n \times k$, and let B be a matrix over K of dimension $\mathrm{width}A \times m$. Then $A \cdot B$ is a matrix over K of dimension $\mathrm{len}A \times \mathrm{width}B$.

Let us consider K, let M be a matrix over K, and let a be an element of K. The functor $a \cdot M$ yields a matrix over K and is defined as follows:

(Def. 5) $\mathrm{len}(a \cdot M) = \mathrm{len}M$ and $\mathrm{width}(a \cdot M) = \mathrm{width}M$ and for all i, j such that $\langle i, j\rangle \in$ the indices of M holds $(a \cdot M) \circ (i, j) = a \cdot (M \circ (i, j))$.

Let us consider K, let M be a matrix over K, and let a be an element of K. The functor $M \cdot a$ yielding a matrix over K is defined as follows:

(Def. 6) $M \cdot a = a \cdot M$.

One can prove the following propositions:

(8) For all finite sequences p, q of elements of the carrier of K such that $\mathrm{len}\,p = \mathrm{len}\,q$ holds $\mathrm{len}(p \bullet q) = \mathrm{len}\,p$ and $\mathrm{len}(p \bullet q) = \mathrm{len}\,q$.

(9) For all i, l such that $\langle i, l\rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ and $l = i$ holds $\mathrm{Line}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}, i)(l) = \mathbf{1}_K$.

(10) For all i, l such that $\langle i, l\rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ and $l \neq i$ holds $\mathrm{Line}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}, i)(l) = 0_K$.

[1] The proposition (2) has been removed.

(11) For all l, j such that $\langle l, j\rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ and $l = j$ holds $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})_{\square,j}(l) = \mathbf{1}_K$.

(12) For all l, j such that $\langle l, j\rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ and $l \neq j$ holds $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})_{\square,j}(l) = 0_K$.

(13) For every add-associative right zeroed right complementable non empty loop structure K holds $\sum(n \mapsto 0_K) = 0_K$.

(14) Let K be an add-associative right zeroed right complementable non empty loop structure, p be a finite sequence of elements of the carrier of K, and given i. If $i \in \operatorname{dom} p$ and for every k such that $k \in \operatorname{dom} p$ and $k \neq i$ holds $p(k) = 0_K$, then $\sum p = p(i)$.

(15) For all finite sequences p, q of elements of the carrier of K holds $\operatorname{len}(p \bullet q) = \min(\operatorname{len} p, \operatorname{len} q)$.

(16) Let p, q be finite sequences of elements of the carrier of K and given i. Suppose $i \in \operatorname{dom} p$ and $p(i) = \mathbf{1}_K$ and for every k such that $k \in \operatorname{dom} p$ and $k \neq i$ holds $p(k) = 0_K$. Let given j such that $j \in \operatorname{dom}(p \bullet q)$. Then

(i) if $i = j$, then $(p \bullet q)(j) = q(i)$, and

(ii) if $i \neq j$, then $(p \bullet q)(j) = 0_K$.

(17) For all i, j such that $\langle i, j\rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ holds if $i = j$, then $\operatorname{Line}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}, i)(j) = \mathbf{1}_K$ and if $i \neq j$, then $\operatorname{Line}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}, i)(j) = 0_K$.

(18) For all i, j such that $\langle i, j\rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ holds if $i = j$, then $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})_{\square,j}(i) = \mathbf{1}_K$ and if $i \neq j$, then $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})_{\square,j}(i) = 0_K$.

(19) Let p, q be finite sequences of elements of the carrier of K and given i. Suppose $i \in \operatorname{dom} p$ and $i \in \operatorname{dom} q$ and $p(i) = \mathbf{1}_K$ and for every k such that $k \in \operatorname{dom} p$ and $k \neq i$ holds $p(k) = 0_K$. Then $\sum(p \bullet q) = q(i)$.

(20) For every matrix A over K of dimension n holds $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n} \cdot A = A$.

(21) For every matrix A over K of dimension n holds $A \cdot \left(\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{array}\right)_K^{n \times n} = A$.

(22) For all elements a, b of K holds $\langle\langle a \rangle\rangle \cdot \langle\langle b \rangle\rangle = \langle\langle a \cdot b \rangle\rangle$.

(23) For all elements a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 of K holds $\left(\begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array}\right) \cdot \left(\begin{array}{cc} a_2 & b_2 \\ c_2 & d_2 \end{array}\right) = \left(\begin{array}{cc} a_1 \cdot a_2 + b_1 \cdot c_2 & a_1 \cdot b_2 + b_1 \cdot d_2 \\ c_1 \cdot a_2 + d_1 \cdot c_2 & c_1 \cdot b_2 + d_1 \cdot d_2 \end{array}\right)$.

(24) For all matrices A, B over K such that $\mathrm{width} A = \mathrm{len} B$ and $\mathrm{width} B \neq 0$ holds $(A \cdot B)^{\mathrm{T}} = B^{\mathrm{T}} \cdot A^{\mathrm{T}}$.

2. The Product of Matrices

Let I, J be non empty sets, let X be an element of $\mathrm{Fin} I$, and let Y be an element of $\mathrm{Fin} J$. Then $[: X, Y :]$ is an element of $\mathrm{Fin} [: I, J :]$.

Let I, J, D be non empty sets, let G be a binary operation on D, let f be a function from I into D, and let g be a function from J into D. Then $G \circ (f, g)$ is a function from $[: I, J :]$ into D.

We now state a number of propositions:

(25) Let I, J, D be non empty sets, F, G be binary operations on D, f be a function from I into D, g be a function from J into D, X be an element of $\mathrm{Fin} I$, and Y be an element of $\mathrm{Fin} J$. Suppose F is commutative and associative but $[: Y, X :] \neq \emptyset$ or F has a unity but G is commutative. Then $F\text{-}\sum_{[: X, Y :]} (G \circ (f, g)) = F\text{-}\sum_{[: Y, X :]} (G \circ (g, f))$.

(26) Let I, J be non empty sets, F, G be binary operations on D, f be a function from I into D, and g be a function from J into D. Suppose F is commutative and associative and has a unity. Let x be an element of I and y be an element of J. Then $F\text{-}\sum_{[: \{x\}, \{y\} :]} (G \circ (f, g)) = F\text{-}\sum_{\{x\}} G^{\circ}(f, F\text{-}\sum_{\{y\}} g)$.

(27) Let I, J be non empty sets, F, G be binary operations on D, f be a function from I into D, g be a function from J into D, X be an element of $\mathrm{Fin} I$, and Y be an element of $\mathrm{Fin} J$. Suppose F is commutative and associative and has a unity and an inverse operation and G is distributive w.r.t. F. Let x be an element of I. Then $F\text{-}\sum_{[: \{x\}, Y :]} (G \circ (f, g)) = F\text{-}\sum_{\{x\}} G^{\circ}(f, F\text{-}\sum_{Y} g)$.

(28) Let I, J be non empty sets, F, G be binary operations on D, f be a function from I into D, g be a function from J into D, X be an element of $\mathrm{Fin} I$, and Y be an element of $\mathrm{Fin} J$. Suppose F is commutative and associative and has a unity and an inverse operation and G is distributive w.r.t. F. Then $F\text{-}\sum_{[: X, Y :]} (G \circ (f, g)) = F\text{-}\sum_{X} G^{\circ}(f, F\text{-}\sum_{Y} g)$.

(29) Let I, J be non empty sets, F, G be binary operations on D, f be a function from I into D, and g be a function from J into D. Suppose F is commutative and associative and has a unity and G is commutative. Let x be an element of I and y be an element of J. Then $F\text{-}\sum_{[: \{x\}, \{y\} :]} (G \circ (f, g)) = F\text{-}\sum_{\{y\}} G^{\circ}(F\text{-}\sum_{\{x\}} f, g)$.

(30) Let I, J be non empty sets, F, G be binary operations on D, f be a function from I into D, g be a function from J into D, X be an element of $\mathrm{Fin} I$, and Y be an element of $\mathrm{Fin} J$. Suppose that

(i) F is commutative and associative and has a unity and an inverse operation, and

(ii) G is distributive w.r.t. F and commutative.

Then $F\text{-}\sum_{[: X, Y :]} (G \circ (f, g)) = F\text{-}\sum_{Y} G^{\circ}(F\text{-}\sum_{X} f, g)$.

(31) Let I, J be non empty sets, F be a binary operation on D, f be a function from $[: I, J :]$ into D, g be a function from I into D, and Y be an element of $\mathrm{Fin} J$. Suppose F is commutative and associative and has a unity and an inverse operation. Let x be an element of I. If for every element i of I holds $g(i) = F\text{-}\sum_{Y} (\mathrm{curry} f)(i)$, then $F\text{-}\sum_{[: \{x\}, Y :]} f = F\text{-}\sum_{\{x\}} g$.

(32) Let I, J be non empty sets, F be a binary operation on D, f be a function from $[: I, J :]$ into D, g be a function from I into D, X be an element of $\mathrm{Fin}\, I$, and Y be an element of $\mathrm{Fin}\, J$. Suppose for every element i of I holds $g(i) = F\text{-}\sum_Y(\mathrm{curry}\, f)(i)$ and F is commutative and associative and has a unity and an inverse operation. Then $F\text{-}\sum_{[: X, Y :]} f = F\text{-}\sum_X g$.

(33) Let I, J be non empty sets, F be a binary operation on D, f be a function from $[: I, J :]$ into D, g be a function from J into D, and X be an element of $\mathrm{Fin}\, I$. Suppose F is commutative and associative and has a unity and an inverse operation. Let y be an element of J. If for every element j of J holds $g(j) = F\text{-}\sum_X(\mathrm{curry}'\, f)(j)$, then $F\text{-}\sum_{[: X, \{y\} :]} f = F\text{-}\sum_{\{y\}} g$.

(34) Let I, J be non empty sets, F be a binary operation on D, f be a function from $[: I, J :]$ into D, g be a function from J into D, X be an element of $\mathrm{Fin}\, I$, and Y be an element of $\mathrm{Fin}\, J$. Suppose for every element j of J holds $g(j) = F\text{-}\sum_X(\mathrm{curry}'\, f)(j)$ and F is commutative and associative and has a unity and an inverse operation. Then $F\text{-}\sum_{[: X, Y :]} f = F\text{-}\sum_Y g$.

(35) For all matrices A, B, C over K such that $\mathrm{width}\, A = \mathrm{len}\, B$ and $\mathrm{width}\, B = \mathrm{len}\, C$ holds $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

3. Determinant

Let us consider n, K, let M be a matrix over K of dimension n, and let p be an element of the permutations of n-element set. The functor p-Path$M$ yielding a finite sequence of elements of the carrier of K is defined by:

(Def. 7) $\mathrm{len}(p\text{-Path}\, M) = n$ and for all i, j such that $i \in \mathrm{dom}(p\text{-Path}\, M)$ and $j = p(i)$ holds $(p\text{-Path}\, M)(i) = M \circ (i, j)$.

Let us consider n, K and let M be a matrix over K of dimension n. The product on paths of M yields a function from the permutations of n-element set into the carrier of K and is defined by the condition (Def. 8).

(Def. 8) Let p be an element of the permutations of n-element set. Then (the product on paths of $M)(p) = (-1)^{\mathrm{sgn}(p)}$(the multiplication of $K \circledast (p\text{-Path}\, M)$).

Let us consider n, let us consider K, and let M be a matrix over K of dimension n. The functor $\mathrm{Det}\, M$ yielding an element of the carrier of K is defined as follows:

(Def. 9) $\mathrm{Det}\, M =$ (the addition of K)-$\sum_{\Omega^{\mathrm{f}}_{\text{the permutations of } n\text{-element set}}}$ (the product on paths of M).

In the sequel a denotes an element of K.

Next we state the proposition

(36) $\mathrm{Det}\langle\langle a\rangle\rangle = a$.

Let us consider n, let us consider K, and let M be a matrix over K of dimension n. The diagonal of M yields a finite sequence of elements of the carrier of K and is defined by:

(Def. 10) len (the diagonal of $M) = n$ and for every i such that $i \in \mathrm{Seg}\, n$ holds (the diagonal of $M)(i) = M \circ (i, i)$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[2] Grzegorz Bancerek. Curried and uncurried functions. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/funct_5.html.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[4] Czesław Byliński. Basic functions and operations on functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_3.html.

[5] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[6] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[7] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[8] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[9] Czesław Byliński. Binary operations applied to finite sequences. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseqop.html.

[10] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[11] Czesław Byliński. Semigroup operations on finite subsets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/setwop_2.html.

[12] Czesław Byliński. Subcategories and products of categories. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/cat_2.html.

[13] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[14] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/matrix_1.html.

[15] Katarzyna Jankowska. Transpose matrices and groups of permutations. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/matrix_2.html.

[16] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/vectsp_1.html.

[17] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funcop_1.html.

[18] Andrzej Trybulec. Semilattice operations on finite subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setwiseo.html.

[19] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[20] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[21] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/square_1.html.

[22] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finsub_1.html.

[23] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[24] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[25] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[26] Katarzyna Zawadzka. Sum and product of finite sequences of elements of a field. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/fvsum_1.html.

Received May 17, 1993

Published January 2, 2004