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Many-Argument Relations

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Summary. Definitions of relations based on finite sequences. The arity of relation, the set of logical values *Boolean* consisting of *false* and *true* and the operations of negation and conjunction on them are defined.

MML Identifier: `MARGREL1`.

WWW: `http://mizar.org/JFM/Vol2/margrel1.html`

The articles [4], [2], [6], [1], [7], [3], and [5] provide the notation and terminology for this paper.

In this paper k is a natural number and D is a non empty set.

Let B, A be non empty sets and let b be an element of B. Then $A \longmapsto b$ is an element of B^A.

Let I_1 be a set. We say that I_1 is relation-like if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) For every set x such that $x \in I_1$ holds x is a finite sequence, and

(ii) for all finite sequences a, b such that $a \in I_1$ and $b \in I_1$ holds $\operatorname{len} a = \operatorname{len} b$.

Let us mention that there exists a set which is relation-like.

A relation is a relation-like set.

We follow the rules: X denotes a set, p, r denote relations, and a, b denote finite sequences.

The following two propositions are true:

(7)[1] If $X \subseteq p$, then X is relation-like.

(8) $\{a\}$ is relation-like.

The scheme *rel exist* deals with a set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists r such that for every a holds $a \in r$ iff $a \in \mathcal{A}$ and $\mathcal{P}[a]$

provided the parameters meet the following condition:

- For all a, b such that $\mathcal{P}[a]$ and $\mathcal{P}[b]$ holds $\operatorname{len} a = \operatorname{len} b$.

Let us consider p, r. Let us observe that $p = r$ if and only if:

(Def. 2) For every a holds $a \in p$ iff $a \in r$.

Let us note that $\emptyset$ is relation-like.

We now state the proposition

(9) For every p such that for every a holds $a \notin p$ holds $p = \emptyset$.

Let us consider p. Let us assume that $p \neq \emptyset$. The functor $\operatorname{Arity}(p)$ yields a natural number and is defined by:

[1] The propositions (1)–(6) have been removed.

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(Def. 4)[2] For every a such that $a \in p$ holds $\mathrm{Arity}(p) = \mathrm{len}\, a$.

Let us consider k. A relation is called a k-ary relation if:

(Def. 5) For every a such that $a \in$ it holds $\mathrm{len}\, a = k$.

Let X be a set. A relation is called a relation on X if:

(Def. 6) For every a such that $a \in$ it holds $\mathrm{rng}\, a \subseteq X$.

Next we state two propositions:

(20)[3] $\emptyset$ is a relation on X.

(21) $\emptyset$ is a k-ary relation.

Let us consider X, k. A relation is called a k-ary relation of X if:

(Def. 7) It is a relation on X and it is a k-ary relation.

Let us consider D. The functor $\mathrm{Rel}(D)$ yielding a set is defined by the condition (Def. 8).

(Def. 8) Let given X. Then $X \in \mathrm{Rel}(D)$ if and only if the following conditions are satisfied:

(i) $X \subseteq D^*$, and

(ii) for all finite sequences a, b of elements of D such that $a \in X$ and $b \in X$ holds $\mathrm{len}\, a = \mathrm{len}\, b$.

Let us consider D. Note that $\mathrm{Rel}(D)$ is non empty.
Let D be a non empty set. A relation on D is an element of $\mathrm{Rel}(D)$.
In the sequel a denotes a finite sequence of elements of D and p, r denote elements of $\mathrm{Rel}(D)$.
Next we state three propositions:

(26)[4] If $X \subseteq r$, then X is an element of $\mathrm{Rel}(D)$.

(27) $\{a\}$ is an element of $\mathrm{Rel}(D)$.

(28) For all elements x, y of D holds $\{\langle x, y \rangle\}$ is an element of $\mathrm{Rel}(D)$.

Let us consider D, p, r. Let us observe that $p = r$ if and only if:

(Def. 9) For every a holds $a \in p$ iff $a \in r$.

The scheme *rel D exist* deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:
There exists an element r of $\mathrm{Rel}(\mathcal{A})$ such that for every finite sequence a of elements of $\mathcal{A}$ holds $a \in r$ iff $\mathcal{P}[a]$
provided the parameters satisfy the following condition:

- For all finite sequences a, b of elements of $\mathcal{A}$ such that $\mathcal{P}[a]$ and $\mathcal{P}[b]$ holds $\mathrm{len}\, a = \mathrm{len}\, b$.

Let us consider D. The functor $\varnothing_D$ yielding an element of $\mathrm{Rel}(D)$ is defined as follows:

(Def. 10) $a \notin \varnothing_D$.

Next we state the proposition

(32)[5] $\varnothing_D = \emptyset$.

Let us consider D, p. Let us assume that $p \neq \varnothing_D$. The functor $\mathrm{Arity}(p)$ yields a natural number and is defined by:

(Def. 11) If $a \in p$, then $\mathrm{Arity}(p) = \mathrm{len}\, a$.

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[2] The definition (Def. 3) has been removed.
[3] The propositions (10)–(19) have been removed.
[4] The propositions (22)–(25) have been removed.
[5] The propositions (29)–(31) have been removed.

The scheme *rel D exist2* deals with a non empty set $\mathcal{A}$, a natural number $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

There exists an element r of $\mathrm{Rel}(\mathcal{A})$ such that for every finite sequence a of elements of $\mathcal{A}$ if $\mathrm{len}\, a = \mathcal{B}$, then $a \in r$ iff $\mathcal{P}[a]$

for all values of the parameters.

The set *Boolean* is defined as follows:

(Def. 12) $Boolean = \{0, 1\}$.

One can verify that *Boolean* is non empty.

The element *false* of *Boolean* is defined as follows:

(Def. 13) $false = 0$.

The element *true* of *Boolean* is defined as follows:

(Def. 14) $true = 1$.

Next we state three propositions:

(36)[6] $false = 0$ and $true = 1$.

(37) $Boolean = \{false, true\}$.

(38) $false \neq true$.

Let x be a set. We say that x is boolean if and only if:

(Def. 15) $x \in Boolean$.

Let us mention that there exists a set which is boolean and every element of *Boolean* is boolean.

In the sequel u, v, w are boolean sets.

Next we state the proposition

(39) $v = false$ or $v = true$.

Let v be a boolean set. The functor $\neg v$ is defined by:

(Def. 16)(i) $\neg v = true$ if $v = false$,

(ii) $\neg v = false$ if $v = true$.

Let w be a boolean set. The functor $v \wedge w$ is defined by:

(Def. 17) $v \wedge w = \begin{cases} \text{(i)} \quad true, \text{ if } v = true \text{ and } w = true, \\ false, \text{ otherwise.} \end{cases}$

Let us observe that the functor $v \wedge w$ is commutative.

Let v be a boolean set. Note that $\neg v$ is boolean. Let w be a boolean set. One can verify that $v \wedge w$ is boolean.

Let v be an element of *Boolean*. Then $\neg v$ is an element of *Boolean*. Let w be an element of *Boolean*. Then $v \wedge w$ is an element of *Boolean*.

One can prove the following propositions:

(40) $\neg\neg v = v$.

(41) $v = false$ iff $\neg v = true$ and $v = true$ iff $\neg v = false$.

(43)[7] $v \neq true$ iff $v = false$.

(45)[8] $v \wedge w = true$ iff $v = true$ and $w = true$ and $v \wedge w = false$ iff $v = false$ or $w = false$.

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[6] The propositions (33)–(35) have been removed.
[7] The proposition (42) has been removed.
[8] The proposition (44) has been removed.

(46) $v \wedge \neg v = \mathit{false}$.

(47) $\neg(v \wedge \neg v) = \mathit{true}$.

(49)[9] $\mathit{false} \wedge v = \mathit{false}$.

(50) $\mathit{true} \wedge v = v$.

(51) If $v \wedge v = \mathit{false}$, then $v = \mathit{false}$.

(52) $v \wedge (w \wedge u) = (v \wedge w) \wedge u$.

Let us consider X. The functor $\mathit{Boolean}(\mathit{false} \notin X)$ is defined by:

(Def. 18) $\mathit{Boolean}(\mathit{false} \notin X) = \begin{cases} \text{(i)} \quad \mathit{true}, & \text{if } \mathit{false} \notin X, \\ \mathit{false}, & \text{otherwise.} \end{cases}$

Let us consider X. One can check that $\mathit{Boolean}(\mathit{false} \notin X)$ is boolean.
Let us consider X. Then $\mathit{Boolean}(\mathit{false} \notin X)$ is an element of *Boolean.*
One can prove the following proposition

(53) $\mathit{false} \notin X$ iff $\mathit{Boolean}(\mathit{false} \notin X) = \mathit{true}$ and $\mathit{false} \in X$ iff $\mathit{Boolean}(\mathit{false} \notin X) = \mathit{false}$.

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[9] The proposition (48) has been removed.