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The Limit of a Real Function at Infinity

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Summary. We introduced the halflines (*open* and *closed*), real sequences divergent to infinity (*plus* and *minus*) and the proper and improper limit of a real function at infinty. We prove basic properties of halflines, sequences divergent to infinity and the limit of function at infinity.

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The articles [11], [14], [1], [12], [2], [9], [5], [3], [4], [8], [15], [13], [6], [10], and [7] provide the notation and terminology for this paper.

For simplicity, we follow the rules: r_1, r_2, g_1, g_2 denote real numbers, n, m, k denote natural numbers, s_1, s_2, s_3 denote sequences of real numbers, and f, f_1, f_2 denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Let us consider n, m. Then $\max(n,m)$ is a natural number.

We now state the proposition

(1) If $0 \leq r_1$ and $r_1 < r_2$ and $0 < g_1$ and $g_1 \leq g_2$, then $r_1 \cdot g_1 < r_2 \cdot g_2$.

Let r be a real number. We introduce $]-\infty, r[$ as a synonym of $\mathrm{HL}(r)$.

In the sequel r, r_1, r_2, g, g_1 denote real numbers.

Let r be a real number. The functor $]-\infty, r]$ yielding a subset of $\mathbb{R}$ is defined as follows:

(Def. 1) $]-\infty, r] = \{g : g \leq r\}$.

The functor $[r, +\infty[$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def. 2) $[r, +\infty[= \{g : r \leq g\}$.

The functor $]r, +\infty[$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def. 3) $]r, +\infty[= \{g : r < g\}$.

One can prove the following propositions:

(8)[1] If $r_1 \leq r_2$, then $]r_2, +\infty[\subseteq]r_1, +\infty[$.

(9) If $r_1 \leq r_2$, then $[r_2, +\infty[\subseteq [r_1, +\infty[$.

(10) $]r, +\infty[\subseteq [r, +\infty[$.

(11) $]r, g[\subseteq]r, +\infty[$.

[1] The propositions (2)–(7) have been removed.

(12) $[r,g] \subseteq [r,+\infty[$.

(13) If $r_1 \leq r_2$, then $]-\infty,r_1[\ \subseteq\ ]-\infty,r_2[$.

(14) If $r_1 \leq r_2$, then $]-\infty,r_1] \subseteq\ ]-\infty,r_2]$.

(15) $]-\infty,r[\ \subseteq\ ]-\infty,r]$.

(16) $]g,r[\ \subseteq\ ]-\infty,r[$.

(17) $[g,r] \subseteq\ ]-\infty,r]$.

(18) $]-\infty,r[\ \cap\ ]g,+\infty[\ =\ ]g,r[$.

(19) $]-\infty,r] \cap [g,+\infty[\ = [g,r]$.

(20) If $r \leq r_1$, then $]r_1,r_2[\ \subseteq\ ]r,+\infty[$ and $[r_1,r_2] \subseteq [r,+\infty[$.

(21) If $r < r_1$, then $[r_1,r_2] \subseteq\ ]r,+\infty[$.

(22) If $r_2 \leq r$, then $]r_1,r_2[\ \subseteq\ ]-\infty,r[$ and $[r_1,r_2] \subseteq\ ]-\infty,r]$.

(23) If $r_2 < r$, then $[r_1,r_2] \subseteq\ ]-\infty,r[$.

(24) $\mathbb{R} \setminus]r,+\infty[\ =\ ]-\infty,r]$ and $\mathbb{R} \setminus [r,+\infty[\ =\ ]-\infty,r[$ and $\mathbb{R} \setminus]-\infty,r[\ = [r,+\infty[$ and $\mathbb{R} \setminus]-\infty,r] =\ ]r,+\infty[$.

(25) $\mathbb{R} \setminus]r_1,r_2[\ =\ ]-\infty,r_1] \cup [r_2,+\infty[$ and $\mathbb{R} \setminus [r_1,r_2] =\ ]-\infty,r_1[\ \cup\ ]r_2,+\infty[$.

(26) If s_1 is non-decreasing, then s_1 is lower bounded and if s_1 is non-increasing, then s_1 is upper bounded.

(27) If s_1 is non-zero and convergent and $\lim s_1 = 0$ and s_1 is non-decreasing, then for every n holds $s_1(n) < 0$.

(28) If s_1 is non-zero and convergent and $\lim s_1 = 0$ and s_1 is non-increasing, then for every n holds $0 < s_1(n)$.

(29) If s_1 is convergent and $0 < \lim s_1$, then there exists n such that for every m such that $n \leq m$ holds $0 < s_1(m)$.

(30) If s_1 is convergent and $0 < \lim s_1$, then there exists n such that for every m such that $n \leq m$ holds $\frac{\lim s_1}{2} < s_1(m)$.

Let us consider s_1. We say that s_1 is divergent to $+\infty$ if and only if:

(Def. 4) For every r there exists n such that for every m such that $n \leq m$ holds $r < s_1(m)$.

We say that s_1 is divergent to $-\infty$ if and only if:

(Def. 5) For every r there exists n such that for every m such that $n \leq m$ holds $s_1(m) < r$.

The following propositions are true:

(33)[2] Suppose s_1 is divergent to $+\infty$ and divergent to $-\infty$. Then there exists n such that for every m such that $n \leq m$ holds $s_1 \uparrow m$ is non-zero.

(34)(i) If $s_1 \uparrow k$ is divergent to $+\infty$, then s_1 is divergent to $+\infty$, and

(ii) if $s_1 \uparrow k$ is divergent to $-\infty$, then s_1 is divergent to $-\infty$.

(35) If s_2 is divergent to $+\infty$ and s_3 is divergent to $+\infty$, then $s_2 + s_3$ is divergent to $+\infty$.

(36) If s_2 is divergent to $+\infty$ and s_3 is lower bounded, then $s_2 + s_3$ is divergent to $+\infty$.

[2] The propositions (31) and (32) have been removed.

(37) If s_2 is divergent to $+\infty$ and s_3 is divergent to $+\infty$, then $s_2\, s_3$ is divergent to $+\infty$.

(38) If s_2 is divergent to $-\infty$ and s_3 is divergent to $-\infty$, then $s_2 + s_3$ is divergent to $-\infty$.

(39) If s_2 is divergent to $-\infty$ and s_3 is upper bounded, then $s_2 + s_3$ is divergent to $-\infty$.

(40)(i) If s_1 is divergent to $+\infty$ and $r > 0$, then $r\, s_1$ is divergent to $+\infty$,
(ii) if s_1 is divergent to $+\infty$ and $r < 0$, then $r\, s_1$ is divergent to $-\infty$, and
(iii) if s_1 is divergent to $+\infty$ and $r = 0$, then $\mathrm{rng}(r\, s_1) = \{0\}$ and $r\, s_1$ is constant.

(41)(i) If s_1 is divergent to $-\infty$ and $r > 0$, then $r\, s_1$ is divergent to $-\infty$,
(ii) if s_1 is divergent to $-\infty$ and $r < 0$, then $r\, s_1$ is divergent to $+\infty$, and
(iii) if s_1 is divergent to $-\infty$ and $r = 0$, then $\mathrm{rng}(r\, s_1) = \{0\}$ and $r\, s_1$ is constant.

(42)(i) If s_1 is divergent to $+\infty$, then $-s_1$ is divergent to $-\infty$, and
(ii) if s_1 is divergent to $-\infty$, then $-s_1$ is divergent to $+\infty$.

(43) If s_1 is lower bounded and s_2 is divergent to $-\infty$, then $s_1 - s_2$ is divergent to $+\infty$.

(44) If s_1 is upper bounded and s_2 is divergent to $+\infty$, then $s_1 - s_2$ is divergent to $-\infty$.

(45) If s_1 is divergent to $+\infty$ and s_2 is convergent, then $s_1 + s_2$ is divergent to $+\infty$.

(46) If s_1 is divergent to $-\infty$ and s_2 is convergent, then $s_1 + s_2$ is divergent to $-\infty$.

(47) If for every n holds $s_1(n) = n$, then s_1 is divergent to $+\infty$.

(48) If for every n holds $s_1(n) = -n$, then s_1 is divergent to $-\infty$.

(49) If s_2 is divergent to $+\infty$ and there exists r such that $r > 0$ and for every n holds $s_3(n) \geq r$, then $s_2\, s_3$ is divergent to $+\infty$.

(50) If s_2 is divergent to $-\infty$ and there exists r such that $0 < r$ and for every n holds $s_3(n) \geq r$, then $s_2\, s_3$ is divergent to $-\infty$.

(51) If s_2 is divergent to $-\infty$ and s_3 is divergent to $-\infty$, then $s_2\, s_3$ is divergent to $+\infty$.

(52) If s_1 is divergent to $+\infty$ and divergent to $-\infty$, then $|s_1|$ is divergent to $+\infty$.

(53) If s_1 is divergent to $+\infty$ and s_2 is a subsequence of s_1, then s_2 is divergent to $+\infty$.

(54) If s_1 is divergent to $-\infty$ and s_2 is a subsequence of s_1, then s_2 is divergent to $-\infty$.

(55) If s_2 is divergent to $+\infty$ and s_3 is convergent and $0 < \lim s_3$, then $s_2\, s_3$ is divergent to $+\infty$.

(56) If s_1 is non-decreasing and s_1 is not upper bounded, then s_1 is divergent to $+\infty$.

(57) If s_1 is non-increasing and s_1 is not lower bounded, then s_1 is divergent to $-\infty$.

(58) If s_1 is increasing and s_1 is not upper bounded, then s_1 is divergent to $+\infty$.

(59) If s_1 is decreasing and s_1 is not lower bounded, then s_1 is divergent to $-\infty$.

(60) If s_1 is monotone, then s_1 is convergent, divergent to $+\infty$, and divergent to $-\infty$.

(61) If s_1 is divergent to $+\infty$ and divergent to $-\infty$, then ${s_1}^{-1}$ is convergent and $\lim({s_1}^{-1}) = 0$.

(62) Suppose s_1 is non-zero and convergent and $\lim s_1 = 0$ and there exists k such that for every n such that $k \leq n$ holds $0 < s_1(n)$. Then ${s_1}^{-1}$ is divergent to $+\infty$.

(63) Suppose s_1 is non-zero and convergent and $\lim s_1 = 0$ and there exists k such that for every n such that $k \leq n$ holds $s_1(n) < 0$. Then ${s_1}^{-1}$ is divergent to $-\infty$.

(64) If s_1 is non-zero and convergent and $\lim s_1 = 0$ and s_1 is non-decreasing, then ${s_1}^{-1}$ is divergent to $-\infty$.

(65) If s_1 is non-zero and convergent and $\lim s_1 = 0$ and s_1 is non-increasing, then ${s_1}^{-1}$ is divergent to $+\infty$.

(66) If s_1 is non-zero and convergent and $\lim s_1 = 0$ and s_1 is increasing, then ${s_1}^{-1}$ is divergent to $-\infty$.

(67) If s_1 is non-zero and convergent and $\lim s_1 = 0$ and s_1 is decreasing, then ${s_1}^{-1}$ is divergent to $+\infty$.

(68) Suppose s_2 is bounded and s_3 is divergent to $+\infty$, divergent to $-\infty$, and non-zero. Then s_2/s_3 is convergent and $\lim(s_2/s_3) = 0$.

(69) If s_1 is divergent to $+\infty$ and for every n holds $s_1(n) \leq s_2(n)$, then s_2 is divergent to $+\infty$.

(70) If s_1 is divergent to $-\infty$ and for every n holds $s_2(n) \leq s_1(n)$, then s_2 is divergent to $-\infty$.

Let us consider f. We say that f is convergent in $+\infty$ if and only if the conditions (Def. 6) are satisfied.

(Def. 6)(i) For every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$, and

(ii) there exists g such that for every s_1 such that s_1 is divergent to $+\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

We say that f is divergent in $+\infty$ to $+\infty$ if and only if the conditions (Def. 7) are satisfied.

(Def. 7)(i) For every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$, and

(ii) for every s_1 such that s_1 is divergent to $+\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is divergent to $+\infty$.

We say that f is divergent in $+\infty$ to $-\infty$ if and only if the conditions (Def. 8) are satisfied.

(Def. 8)(i) For every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$, and

(ii) for every s_1 such that s_1 is divergent to $+\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is divergent to $-\infty$.

We say that f is convergent in $-\infty$ if and only if the conditions (Def. 9) are satisfied.

(Def. 9)(i) For every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$, and

(ii) there exists g such that for every s_1 such that s_1 is divergent to $-\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

We say that f is divergent in $-\infty$ to $+\infty$ if and only if the conditions (Def. 10) are satisfied.

(Def. 10)(i) For every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$, and

(ii) for every s_1 such that s_1 is divergent to $-\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is divergent to $+\infty$.

We say that f is divergent in $-\infty$ to $-\infty$ if and only if the conditions (Def. 11) are satisfied.

(Def. 11)(i) For every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$, and

(ii) for every s_1 such that s_1 is divergent to $-\infty$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ holds $f \cdot s_1$ is divergent to $-\infty$.

Next we state a number of propositions:

(77)[3] f is convergent in $+\infty$ if and only if the following conditions are satisfied:

(i) for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$, and

(ii) there exists g such that for every g_1 such that $0 < g_1$ there exists r such that for every r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$.

(78) f is convergent in $-\infty$ if and only if the following conditions are satisfied:

(i) for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$, and

(ii) there exists g such that for every g_1 such that $0 < g_1$ there exists r such that for every r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$.

(79) f is divergent in $+\infty$ to $+\infty$ if and only if for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and for every g there exists r such that for every r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $g < f(r_1)$.

(80) f is divergent in $+\infty$ to $-\infty$ if and only if for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and for every g there exists r such that for every r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < g$.

(81) f is divergent in $-\infty$ to $+\infty$ if and only if for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and for every g there exists r such that for every r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} f$ holds $g < f(r_1)$.

(82) f is divergent in $-\infty$ to $-\infty$ if and only if for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and for every g there exists r such that for every r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < g$.

(83) Suppose that

(i) f_1 is divergent in $+\infty$ to $+\infty$,

(ii) f_2 is divergent in $+\infty$ to $+\infty$, and

(iii) for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$.

Then $f_1 + f_2$ is divergent in $+\infty$ to $+\infty$ and $f_1\ f_2$ is divergent in $+\infty$ to $+\infty$.

(84) Suppose that

(i) f_1 is divergent in $+\infty$ to $-\infty$,

(ii) f_2 is divergent in $+\infty$ to $-\infty$, and

(iii) for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$.

Then $f_1 + f_2$ is divergent in $+\infty$ to $-\infty$ and $f_1\ f_2$ is divergent in $+\infty$ to $+\infty$.

(85) Suppose that

(i) f_1 is divergent in $-\infty$ to $+\infty$,

(ii) f_2 is divergent in $-\infty$ to $+\infty$, and

(iii) for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$.

Then $f_1 + f_2$ is divergent in $-\infty$ to $+\infty$ and $f_1\ f_2$ is divergent in $-\infty$ to $+\infty$.

(86) Suppose that

(i) f_1 is divergent in $-\infty$ to $-\infty$,

(ii) f_2 is divergent in $-\infty$ to $-\infty$, and

(iii) for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$.

Then $f_1 + f_2$ is divergent in $-\infty$ to $-\infty$ and $f_1\ f_2$ is divergent in $-\infty$ to $+\infty$.

[3] The propositions (71)–(76) have been removed.

(87) Suppose that

(i) f_1 is divergent in $+\infty$ to $+\infty$,

(ii) for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_1 + f_2)$, and

(iii) there exists r such that f_2 is lower bounded on $]r, +\infty[$.

Then $f_1 + f_2$ is divergent in $+\infty$ to $+\infty$.

(88) Suppose that

(i) f_1 is divergent in $+\infty$ to $+\infty$,

(ii) for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_1\ f_2)$, and

(iii) there exist r, r_1 such that $0 < r$ and for every g such that $g \in \mathrm{dom} f_2 \cap]r_1, +\infty[$ holds $r \leq f_2(g)$.

Then $f_1\ f_2$ is divergent in $+\infty$ to $+\infty$.

(89) Suppose that

(i) f_1 is divergent in $-\infty$ to $+\infty$,

(ii) for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_1 + f_2)$, and

(iii) there exists r such that f_2 is lower bounded on $]-\infty, r[$.

Then $f_1 + f_2$ is divergent in $-\infty$ to $+\infty$.

(90) Suppose that

(i) f_1 is divergent in $-\infty$ to $+\infty$,

(ii) for every r there exists g such that $g < r$ and $g \in \mathrm{dom}(f_1\ f_2)$, and

(iii) there exist r, r_1 such that $0 < r$ and for every g such that $g \in \mathrm{dom} f_2 \cap]-\infty, r_1[$ holds $r \leq f_2(g)$.

Then $f_1\ f_2$ is divergent in $-\infty$ to $+\infty$.

(91)(i) If f is divergent in $+\infty$ to $+\infty$ and $r > 0$, then $r\, f$ is divergent in $+\infty$ to $+\infty$,

(ii) if f is divergent in $+\infty$ to $+\infty$ and $r < 0$, then $r\, f$ is divergent in $+\infty$ to $-\infty$,

(iii) if f is divergent in $+\infty$ to $-\infty$ and $r > 0$, then $r\, f$ is divergent in $+\infty$ to $-\infty$, and

(iv) if f is divergent in $+\infty$ to $-\infty$ and $r < 0$, then $r\, f$ is divergent in $+\infty$ to $+\infty$.

(92)(i) If f is divergent in $-\infty$ to $+\infty$ and $r > 0$, then $r\, f$ is divergent in $-\infty$ to $+\infty$,

(ii) if f is divergent in $-\infty$ to $+\infty$ and $r < 0$, then $r\, f$ is divergent in $-\infty$ to $-\infty$,

(iii) if f is divergent in $-\infty$ to $-\infty$ and $r > 0$, then $r\, f$ is divergent in $-\infty$ to $-\infty$, and

(iv) if f is divergent in $-\infty$ to $-\infty$ and $r < 0$, then $r\, f$ is divergent in $-\infty$ to $+\infty$.

(93) Suppose f is divergent in $+\infty$ to $+\infty$ and divergent in $+\infty$ to $-\infty$. Then $|f|$ is divergent in $+\infty$ to $+\infty$.

(94) Suppose f is divergent in $-\infty$ to $+\infty$ and divergent in $-\infty$ to $-\infty$. Then $|f|$ is divergent in $-\infty$ to $+\infty$.

(95) Suppose there exists r such that f is non-decreasing on $]r, +\infty[$ and f is not upper bounded on $]r, +\infty[$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom} f$. Then f is divergent in $+\infty$ to $+\infty$.

(96) Suppose there exists r such that f is increasing on $]r, +\infty[$ and f is not upper bounded on $]r, +\infty[$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom} f$. Then f is divergent in $+\infty$ to $+\infty$.

(97) Suppose there exists r such that f is non increasing on $]r, +\infty[$ and f is not lower bounded on $]r, +\infty[$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom} f$. Then f is divergent in $+\infty$ to $-\infty$.

(98) Suppose there exists r such that f is decreasing on $]r,+\infty[$ and f is not lower bounded on $]r,+\infty[$ and for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$. Then f is divergent in $+\infty$ to $-\infty$.

(99) Suppose there exists r such that f is non increasing on $]-\infty,r[$ and f is not upper bounded on $]-\infty,r[$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$. Then f is divergent in $-\infty$ to $+\infty$.

(100) Suppose there exists r such that f is decreasing on $]-\infty,r[$ and f is not upper bounded on $]-\infty,r[$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$. Then f is divergent in $-\infty$ to $+\infty$.

(101) Suppose there exists r such that f is non-decreasing on $]-\infty,r[$ and f is not lower bounded on $]-\infty,r[$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$. Then f is divergent in $-\infty$ to $-\infty$.

(102) Suppose there exists r such that f is increasing on $]-\infty,r[$ and f is not lower bounded on $]-\infty,r[$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$. Then f is divergent in $-\infty$ to $-\infty$.

(103) Suppose that

(i) f_1 is divergent in $+\infty$ to $+\infty$,

(ii) for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$, and

(iii) there exists r such that $\operatorname{dom} f \cap]r,+\infty[\subseteq \operatorname{dom} f_1 \cap]r,+\infty[$ and for every g such that $g \in \operatorname{dom} f \cap]r,+\infty[$ holds $f_1(g) \leq f(g)$.

Then f is divergent in $+\infty$ to $+\infty$.

(104) Suppose that

(i) f_1 is divergent in $+\infty$ to $-\infty$,

(ii) for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$, and

(iii) there exists r such that $\operatorname{dom} f \cap]r,+\infty[\subseteq \operatorname{dom} f_1 \cap]r,+\infty[$ and for every g such that $g \in \operatorname{dom} f \cap]r,+\infty[$ holds $f(g) \leq f_1(g)$.

Then f is divergent in $+\infty$ to $-\infty$.

(105) Suppose that

(i) f_1 is divergent in $-\infty$ to $+\infty$,

(ii) for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$, and

(iii) there exists r such that $\operatorname{dom} f \cap]-\infty,r[\subseteq \operatorname{dom} f_1 \cap]-\infty,r[$ and for every g such that $g \in \operatorname{dom} f \cap]-\infty,r[$ holds $f_1(g) \leq f(g)$.

Then f is divergent in $-\infty$ to $+\infty$.

(106) Suppose that

(i) f_1 is divergent in $-\infty$ to $-\infty$,

(ii) for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$, and

(iii) there exists r such that $\operatorname{dom} f \cap]-\infty,r[\subseteq \operatorname{dom} f_1 \cap]-\infty,r[$ and for every g such that $g \in \operatorname{dom} f \cap]-\infty,r[$ holds $f(g) \leq f_1(g)$.

Then f is divergent in $-\infty$ to $-\infty$.

(107) Suppose f_1 is divergent in $+\infty$ to $+\infty$ and there exists r such that $]r,+\infty[\subseteq \operatorname{dom} f \cap \operatorname{dom} f_1$ and for every g such that $g \in]r,+\infty[$ holds $f_1(g) \leq f(g)$. Then f is divergent in $+\infty$ to $+\infty$.

(108) Suppose f_1 is divergent in $+\infty$ to $-\infty$ and there exists r such that $]r,+\infty[\subseteq \operatorname{dom} f \cap \operatorname{dom} f_1$ and for every g such that $g \in]r,+\infty[$ holds $f(g) \leq f_1(g)$. Then f is divergent in $+\infty$ to $-\infty$.

(109) Suppose f_1 is divergent in $-\infty$ to $+\infty$ and there exists r such that $]-\infty, r[\, \subseteq \mathrm{dom}\, f \cap \mathrm{dom}\, f_1$ and for every g such that $g \in \,]-\infty, r[$ holds $f_1(g) \leq f(g)$. Then f is divergent in $-\infty$ to $+\infty$.

(110) Suppose f_1 is divergent in $-\infty$ to $-\infty$ and there exists r such that $]-\infty, r[\, \subseteq \mathrm{dom}\, f \cap \mathrm{dom}\, f_1$ and for every g such that $g \in \,]-\infty, r[$ holds $f(g) \leq f_1(g)$. Then f is divergent in $-\infty$ to $-\infty$.

Let us consider f. Let us assume that f is convergent in $+\infty$. The functor $\lim_{+\infty} f$ yields a real number and is defined by:

(Def. 12) For every s_1 such that s_1 is divergent to $+\infty$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = \lim_{+\infty} f$.

Let us consider f. Let us assume that f is convergent in $-\infty$. The functor $\lim_{-\infty} f$ yields a real number and is defined by:

(Def. 13) For every s_1 such that s_1 is divergent to $-\infty$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = \lim_{-\infty} f$.

We now state a number of propositions:

(113)[4] Suppose f is convergent in $-\infty$. Then $\lim_{-\infty} f = g$ if and only if for every g_1 such that $0 < g_1$ there exists r such that for every r_1 such that $r_1 < r$ and $r_1 \in \mathrm{dom}\, f$ holds $|f(r_1) - g| < g_1$.

(114) Suppose f is convergent in $+\infty$. Then $\lim_{+\infty} f = g$ if and only if for every g_1 such that $0 < g_1$ there exists r such that for every r_1 such that $r < r_1$ and $r_1 \in \mathrm{dom}\, f$ holds $|f(r_1) - g| < g_1$.

(115) If f is convergent in $+\infty$, then $r\, f$ is convergent in $+\infty$ and $\lim_{+\infty}(r\, f) = r \cdot \lim_{+\infty} f$.

(116) If f is convergent in $+\infty$, then $-f$ is convergent in $+\infty$ and $\lim_{+\infty}(-f) = -\lim_{+\infty} f$.

(117) Suppose f_1 is convergent in $+\infty$ and f_2 is convergent in $+\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_1 + f_2)$. Then $f_1 + f_2$ is convergent in $+\infty$ and $\lim_{+\infty}(f_1 + f_2) = \lim_{+\infty} f_1 + \lim_{+\infty} f_2$.

(118) Suppose f_1 is convergent in $+\infty$ and f_2 is convergent in $+\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_1 - f_2)$. Then $f_1 - f_2$ is convergent in $+\infty$ and $\lim_{+\infty}(f_1 - f_2) = \lim_{+\infty} f_1 - \lim_{+\infty} f_2$.

(119) If f is convergent in $+\infty$ and $f^{-1}(\{0\}) = \emptyset$ and $\lim_{+\infty} f \neq 0$, then $\frac{1}{f}$ is convergent in $+\infty$ and $\lim_{+\infty}(\frac{1}{f}) = (\lim_{+\infty} f)^{-1}$.

(120) If f is convergent in $+\infty$, then $|f|$ is convergent in $+\infty$ and $\lim_{+\infty}|f| = |\lim_{+\infty} f|$.

(121) Suppose f is convergent in $+\infty$ and $\lim_{+\infty} f \neq 0$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}\, f$ and $f(g) \neq 0$. Then $\frac{1}{f}$ is convergent in $+\infty$ and $\lim_{+\infty}(\frac{1}{f}) = (\lim_{+\infty} f)^{-1}$.

(122) Suppose f_1 is convergent in $+\infty$ and f_2 is convergent in $+\infty$ and for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(f_1\, f_2)$. Then $f_1\, f_2$ is convergent in $+\infty$ and $\lim_{+\infty}(f_1\, f_2) = \lim_{+\infty} f_1 \cdot \lim_{+\infty} f_2$.

(123) Suppose that

(i) f_1 is convergent in $+\infty$,

(ii) f_2 is convergent in $+\infty$,

(iii) $\lim_{+\infty} f_2 \neq 0$, and

(iv) for every r there exists g such that $r < g$ and $g \in \mathrm{dom}(\frac{f_1}{f_2})$.

Then $\frac{f_1}{f_2}$ is convergent in $+\infty$ and $\lim_{+\infty}(\frac{f_1}{f_2}) = \frac{\lim_{+\infty} f_1}{\lim_{+\infty} f_2}$.

(124) If f is convergent in $-\infty$, then $r\, f$ is convergent in $-\infty$ and $\lim_{-\infty}(r\, f) = r \cdot \lim_{-\infty} f$.

[4] The propositions (111) and (112) have been removed.

(125) If f is convergent in $-\infty$, then $-f$ is convergent in $-\infty$ and $\lim_{-\infty}(-f) = -\lim_{-\infty} f$.

(126) Suppose f_1 is convergent in $-\infty$ and f_2 is convergent in $-\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom}(f_1 + f_2)$. Then $f_1 + f_2$ is convergent in $-\infty$ and $\lim_{-\infty}(f_1 + f_2) = \lim_{-\infty} f_1 + \lim_{-\infty} f_2$.

(127) Suppose f_1 is convergent in $-\infty$ and f_2 is convergent in $-\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom}(f_1 - f_2)$. Then $f_1 - f_2$ is convergent in $-\infty$ and $\lim_{-\infty}(f_1 - f_2) = \lim_{-\infty} f_1 - \lim_{-\infty} f_2$.

(128) If f is convergent in $-\infty$ and $f^{-1}(\{0\}) = \emptyset$ and $\lim_{-\infty} f \neq 0$, then $\frac{1}{f}$ is convergent in $-\infty$ and $\lim_{-\infty}(\frac{1}{f}) = (\lim_{-\infty} f)^{-1}$.

(129) If f is convergent in $-\infty$, then $|f|$ is convergent in $-\infty$ and $\lim_{-\infty}|f| = |\lim_{-\infty} f|$.

(130) Suppose f is convergent in $-\infty$ and $\lim_{-\infty} f \neq 0$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$. Then $\frac{1}{f}$ is convergent in $-\infty$ and $\lim_{-\infty}(\frac{1}{f}) = (\lim_{-\infty} f)^{-1}$.

(131) Suppose f_1 is convergent in $-\infty$ and f_2 is convergent in $-\infty$ and for every r there exists g such that $g < r$ and $g \in \operatorname{dom}(f_1\, f_2)$. Then $f_1\, f_2$ is convergent in $-\infty$ and $\lim_{-\infty}(f_1\, f_2) = \lim_{-\infty} f_1 \cdot \lim_{-\infty} f_2$.

(132) Suppose that

(i) f_1 is convergent in $-\infty$,

(ii) f_2 is convergent in $-\infty$,

(iii) $\lim_{-\infty} f_2 \neq 0$, and

(iv) for every r there exists g such that $g < r$ and $g \in \operatorname{dom}(\frac{f_1}{f_2})$.

Then $\frac{f_1}{f_2}$ is convergent in $-\infty$ and $\lim_{-\infty}(\frac{f_1}{f_2}) = \frac{\lim_{-\infty} f_1}{\lim_{-\infty} f_2}$.

(133) Suppose that

(i) f_1 is convergent in $+\infty$,

(ii) $\lim_{+\infty} f_1 = 0$,

(iii) for every r there exists g such that $r < g$ and $g \in \operatorname{dom}(f_1\, f_2)$, and

(iv) there exists r such that f_2 is bounded on $]r, +\infty[$.

Then $f_1\, f_2$ is convergent in $+\infty$ and $\lim_{+\infty}(f_1\, f_2) = 0$.

(134) Suppose that

(i) f_1 is convergent in $-\infty$,

(ii) $\lim_{-\infty} f_1 = 0$,

(iii) for every r there exists g such that $g < r$ and $g \in \operatorname{dom}(f_1\, f_2)$, and

(iv) there exists r such that f_2 is bounded on $]-\infty, r[$.

Then $f_1\, f_2$ is convergent in $-\infty$ and $\lim_{-\infty}(f_1\, f_2) = 0$.

(135) Suppose that

(i) f_1 is convergent in $+\infty$,

(ii) f_2 is convergent in $+\infty$,

(iii) $\lim_{+\infty} f_1 = \lim_{+\infty} f_2$,

(iv) for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$, and

(v) there exists r such that $\operatorname{dom} f_1 \cap]r, +\infty[\subseteq \operatorname{dom} f_2 \cap]r, +\infty[$ and $\operatorname{dom} f \cap]r, +\infty[\subseteq \operatorname{dom} f_1 \cap]r, +\infty[$ or $\operatorname{dom} f_2 \cap]r, +\infty[\subseteq \operatorname{dom} f_1 \cap]r, +\infty[$ and $\operatorname{dom} f \cap]r, +\infty[\subseteq \operatorname{dom} f_2 \cap]r, +\infty[$ but for every g such that $g \in \operatorname{dom} f \cap]r, +\infty[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$.

Then f is convergent in $+\infty$ and $\lim_{+\infty} f = \lim_{+\infty} f_1$.

(136) Suppose that

(i) f_1 is convergent in $+\infty$,

(ii) f_2 is convergent in $+\infty$,

(iii) $\lim_{+\infty} f_1 = \lim_{+\infty} f_2$, and

(iv) there exists r such that $]r,+\infty[\subseteq \operatorname{dom} f_1 \cap \operatorname{dom} f_2 \cap \operatorname{dom} f$ and for every g such that $g \in]r,+\infty[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$.

Then f is convergent in $+\infty$ and $\lim_{+\infty} f = \lim_{+\infty} f_1$.

(137) Suppose that

(i) f_1 is convergent in $-\infty$,

(ii) f_2 is convergent in $-\infty$,

(iii) $\lim_{-\infty} f_1 = \lim_{-\infty} f_2$,

(iv) for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$, and

(v) there exists r such that $\operatorname{dom} f_1 \cap]-\infty,r[\subseteq \operatorname{dom} f_2 \cap]-\infty,r[$ and $\operatorname{dom} f \cap]-\infty,r[\subseteq \operatorname{dom} f_1 \cap]-\infty,r[$ or $\operatorname{dom} f_2 \cap]-\infty,r[\subseteq \operatorname{dom} f_1 \cap]-\infty,r[$ and $\operatorname{dom} f \cap]-\infty,r[\subseteq \operatorname{dom} f_2 \cap]-\infty,r[$ but for every g such that $g \in \operatorname{dom} f \cap]-\infty,r[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$.

Then f is convergent in $-\infty$ and $\lim_{-\infty} f = \lim_{-\infty} f_1$.

(138) Suppose that

(i) f_1 is convergent in $-\infty$,

(ii) f_2 is convergent in $-\infty$,

(iii) $\lim_{-\infty} f_1 = \lim_{-\infty} f_2$, and

(iv) there exists r such that $]-\infty,r[\subseteq \operatorname{dom} f_1 \cap \operatorname{dom} f_2 \cap \operatorname{dom} f$ and for every g such that $g \in]-\infty,r[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$.

Then f is convergent in $-\infty$ and $\lim_{-\infty} f = \lim_{-\infty} f_1$.

(139) Suppose that

(i) f_1 is convergent in $+\infty$,

(ii) f_2 is convergent in $+\infty$, and

(iii) there exists r such that $\operatorname{dom} f_1 \cap]r,+\infty[\subseteq \operatorname{dom} f_2 \cap]r,+\infty[$ and for every g such that $g \in \operatorname{dom} f_1 \cap]r,+\infty[$ holds $f_1(g) \leq f_2(g)$ or $\operatorname{dom} f_2 \cap]r,+\infty[\subseteq \operatorname{dom} f_1 \cap]r,+\infty[$ and for every g such that $g \in \operatorname{dom} f_2 \cap]r,+\infty[$ holds $f_1(g) \leq f_2(g)$.

Then $\lim_{+\infty} f_1 \leq \lim_{+\infty} f_2$.

(140) Suppose that

(i) f_1 is convergent in $-\infty$,

(ii) f_2 is convergent in $-\infty$, and

(iii) there exists r such that $\operatorname{dom} f_1 \cap]-\infty,r[\subseteq \operatorname{dom} f_2 \cap]-\infty,r[$ and for every g such that $g \in \operatorname{dom} f_1 \cap]-\infty,r[$ holds $f_1(g) \leq f_2(g)$ or $\operatorname{dom} f_2 \cap]-\infty,r[\subseteq \operatorname{dom} f_1 \cap]-\infty,r[$ and for every g such that $g \in \operatorname{dom} f_2 \cap]-\infty,r[$ holds $f_1(g) \leq f_2(g)$.

Then $\lim_{-\infty} f_1 \leq \lim_{-\infty} f_2$.

(141) Suppose that

(i) f is divergent in $+\infty$ to $+\infty$ and divergent in $+\infty$ to $-\infty$, and

(ii) for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$.

Then $\frac{1}{f}$ is convergent in $+\infty$ and $\lim_{+\infty}(\frac{1}{f}) = 0$.

(142) Suppose that

(i) f is divergent in $-\infty$ to $+\infty$ and divergent in $-\infty$ to $-\infty$, and

(ii) for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$.

Then $\frac{1}{f}$ is convergent in $-\infty$ and $\lim_{-\infty}(\frac{1}{f}) = 0$.

(143) Suppose that

(i) f is convergent in $+\infty$,

(ii) $\lim_{+\infty} f = 0$,

(iii) for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$, and

(iv) there exists r such that for every g such that $g \in \operatorname{dom} f \cap]r, +\infty[$ holds $0 \leq f(g)$.

Then $\frac{1}{f}$ is divergent in $+\infty$ to $+\infty$.

(144) Suppose that

(i) f is convergent in $+\infty$,

(ii) $\lim_{+\infty} f = 0$,

(iii) for every r there exists g such that $r < g$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$, and

(iv) there exists r such that for every g such that $g \in \operatorname{dom} f \cap]r, +\infty[$ holds $f(g) \leq 0$.

Then $\frac{1}{f}$ is divergent in $+\infty$ to $-\infty$.

(145) Suppose that

(i) f is convergent in $-\infty$,

(ii) $\lim_{-\infty} f = 0$,

(iii) for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$, and

(iv) there exists r such that for every g such that $g \in \operatorname{dom} f \cap]-\infty, r[$ holds $0 \leq f(g)$.

Then $\frac{1}{f}$ is divergent in $-\infty$ to $+\infty$.

(146) Suppose that

(i) f is convergent in $-\infty$,

(ii) $\lim_{-\infty} f = 0$,

(iii) for every r there exists g such that $g < r$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$, and

(iv) there exists r such that for every g such that $g \in \operatorname{dom} f \cap]-\infty, r[$ holds $f(g) \leq 0$.

Then $\frac{1}{f}$ is divergent in $-\infty$ to $-\infty$.

(147) Suppose f is convergent in $+\infty$ and $\lim_{+\infty} f = 0$ and there exists r such that for every g such that $g \in \operatorname{dom} f \cap]r, +\infty[$ holds $0 < f(g)$. Then $\frac{1}{f}$ is divergent in $+\infty$ to $+\infty$.

(148) Suppose f is convergent in $+\infty$ and $\lim_{+\infty} f = 0$ and there exists r such that for every g such that $g \in \operatorname{dom} f \cap]r, +\infty[$ holds $f(g) < 0$. Then $\frac{1}{f}$ is divergent in $+\infty$ to $-\infty$.

(149) Suppose f is convergent in $-\infty$ and $\lim_{-\infty} f = 0$ and there exists r such that for every g such that $g \in \operatorname{dom} f \cap]-\infty, r[$ holds $0 < f(g)$. Then $\frac{1}{f}$ is divergent in $-\infty$ to $+\infty$.

(150) Suppose f is convergent in $-\infty$ and $\lim_{-\infty} f = 0$ and there exists r such that for every g such that $g \in \operatorname{dom} f \cap]-\infty, r[$ holds $f(g) < 0$. Then $\frac{1}{f}$ is divergent in $-\infty$ to $-\infty$.

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