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Lattice of Fuzzy Sets[1]

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Summary. This article concerns a connection of fuzzy logic and lattice theory. Namely, the fuzzy sets form a Heyting lattice with union and intersection of fuzzy sets as meet and join operations. The lattice of fuzzy sets is defined as the product of interval posets. As the final result, we have characterized the composition of fuzzy relations in terms of lattice theory and proved its associativity.

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The articles [17], [9], [23], [6], [7], [16], [1], [8], [22], [19], [20], [15], [24], [21], [14], [18], [2], [3], [4], [12], [10], [5], [13], and [11] provide the notation and terminology for this paper.

1. Posets of Real Numbers

Let R be a relational structure. We say that R is real if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) The carrier of $R \subseteq \mathbb{R}$, and

(ii) for all real numbers x, y such that $x \in$ the carrier of R and $y \in$ the carrier of R holds $\langle x, y\rangle \in$ the internal relation of R iff $x \leq y$.

Let R be a relational structure. We say that R is interval if and only if:

(Def. 2) R is real and there exist real numbers a, b such that $a \leq b$ and the carrier of $R = [a,b]$.

One can check that every relational structure which is interval is also real and non empty.

Let us note that every relational structure which is empty is also real.

One can prove the following proposition

(1) For every subset X of $\mathbb{R}$ there exists a strict relational structure R such that the carrier of $R = X$ and R is real.

One can check that there exists a relational structure which is interval and strict.

We now state the proposition

(2) Let R_1, R_2 be real relational structures. Suppose the carrier of $R_1 =$ the carrier of R_2. Then the relational structure of $R_1 =$ the relational structure of R_2.

Let R be a non empty real relational structure. One can check that every element of R is real.

Let X be a subset of $\mathbb{R}$. The functor RealPoset$X$ yields a real strict relational structure and is defined by:

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(Def. 3) The carrier of $\mathrm{RealPoset}X = X$.

Let X be a non empty subset of $\mathbb{R}$. Note that $\mathrm{RealPoset}X$ is non empty.

Let R be a relational structure and let x, y be elements of R. We introduce $x \preceq y$ and $y \succeq x$ as synonyms of $x \leq y$.

Let x, y be real numbers. We introduce $x \leq_{\mathbb{R}} y$ and $y \geq_{\mathbb{R}} x$ as synonyms of $x \leq y$. We introduce $y <_{\mathbb{R}} x$ and $x >_{\mathbb{R}} y$ as antonyms of $x \leq y$.

The following proposition is true

(3) For every non empty real relational structure R and for all elements x, y of R holds $x \leq_{\mathbb{R}} y$ iff $x \preceq y$.

Let us note that every relational structure which is real is also reflexive, antisymmetric, and transitive.

Let us note that every real non empty relational structure is connected.

Let R be a non empty real relational structure and let x, y be elements of R. Then $\max(x,y)$ is an element of R.

Let R be a non empty real relational structure and let x, y be elements of R. Then $\min(x,y)$ is an element of R.

Let us mention that every real non empty relational structure has l.u.b.'s and g.l.b.'s.

We adopt the following rules: x, y denote real numbers, R denotes a real non empty relational structure, and a, b denote elements of R.

The following four propositions are true:

(4) $a \sqcup b = \max(a,b)$.

(5) $a \sqcap b = \min(a,b)$.

(6) There exists x such that $x \in$ the carrier of R and for every y such that $y \in$ the carrier of R holds $x \leq y$ if and only if R is lower-bounded.

(7) There exists x such that $x \in$ the carrier of R and for every y such that $y \in$ the carrier of R holds $x \geq y$ if and only if R is upper-bounded.

Let us observe that every non empty relational structure which is interval is also bounded.

We now state the proposition

(8) For every interval non empty relational structure R and for every set X holds $\sup X$ exists in R.

Let us note that every interval non empty relational structure is complete.

One can check that every chain is distributive.

One can verify that every interval non empty relational structure is Heyting.

Let us mention that $[0,1]$ is non empty.

Let us mention that $\mathrm{RealPoset}[0,1]$ is interval.

2. Product of Heyting Lattices

One can prove the following propositions:

(9) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a sup-semilattice. Then $\prod J$ has l.u.b.'s.

(10) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a semilattice. Then $\prod J$ has g.l.b.'s.

(11) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a semilattice. Let f, g be elements of $\prod J$ and i be an element of I. Then $(f \sqcap g)(i) = f(i) \sqcap g(i)$.

(12) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a sup-semilattice. Let f, g be elements of $\prod J$ and i be an element of I. Then $(f \sqcup g)(i) = f(i) \sqcup g(i)$.

(13) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a Heyting complete lattice. Then $\prod J$ is complete and Heyting.

Let A be a non empty set and let R be a complete Heyting lattice. Observe that R^A is Heyting.

3. Lattice of Fuzzy Sets

Let A be a non empty set. The functor FuzzyLattice$A$ yields a Heyting complete lattice and is defined by:

(Def. 4) FuzzyLattice$A = (\mathrm{RealPoset}[0,1])^A$.

The following proposition is true

(14) For every non empty set A holds the carrier of FuzzyLattice$A = [0,1]^A$.

Let A be a non empty set. One can check that FuzzyLattice$A$ is constituted functions.

We now state the proposition

(15) Let R be a complete Heyting lattice, X be a subset of R, and y be an element of R. Then $\bigsqcup_R X \sqcap y = \bigsqcup_R \{x \sqcap y; x$ ranges over elements of R: $x \in X\}$.

Let X be a non empty set and let a be an element of FuzzyLattice$X$. The functor ${}^{@}a$ yielding a membership function of X is defined as follows:

(Def. 5) ${}^{@}a = a$.

Let X be a non empty set and let f be a membership function of X. The functor $f^{@}$ yields an element of FuzzyLattice$X$ and is defined as follows:

(Def. 6) $f^{@} = f$.

Let X be a non empty set, let f be a membership function of X, and let x be an element of X. Then $f(x)$ is an element of RealPoset$[0,1]$.

Let X be a non empty set, let f be an element of FuzzyLattice$X$, and let x be an element of X. Then $f(x)$ is an element of RealPoset$[0,1]$.

For simplicity, we use the following convention: C is a non empty set, c is an element of C, f, g are membership functions of C, and s, t are elements of FuzzyLattice$C$.

Next we state several propositions:

(16) For every c holds $f(c) \leq_{\mathbb{R}} g(c)$ iff $f^{@} \preceq g^{@}$.

(17) $s \preceq t$ iff for every c holds $({}^{@}s)(c) \leq_{\mathbb{R}} ({}^{@}t)(c)$.

(18) $\max(f,g) = f^{@} \sqcup g^{@}$.

(19) $s \sqcup t = \max({}^{@}s, {}^{@}t)$.

(20) $\min(f,g) = f^{@} \sqcap g^{@}$.

(21) $s \sqcap t = \min({}^{@}s, {}^{@}t)$.

4. Associativity of Composition of Fuzzy Relations

In this article we present several logical schemes. The scheme *SupDistributivity* deals with a complete lattice $\mathcal{A}$, non empty sets $\mathcal{B}$, $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}}\{\mathcal{F}(x,y); y$ ranges over elements of $\mathcal{C} : \mathcal{Q}[y]\}; x$ ranges over elements of $\mathcal{B} : \mathcal{P}[x]\} = \bigsqcup_{\mathcal{A}}\{\mathcal{F}(x,y); x$ ranges over elements of $\mathcal{B}, y$ ranges over elements of $\mathcal{C} : \mathcal{P}[x] \wedge \mathcal{Q}[y]\}$

for all values of the parameters.

The scheme *SupDistributivity'* deals with a complete lattice $\mathcal{A}$, non empty sets $\mathcal{B}$, $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}}\{\mathcal{F}(x,y); x$ ranges over elements of $\mathcal{B} : \mathcal{P}[x]\}; y$ ranges over elements of $\mathcal{C} : \mathcal{Q}[y]\} = \bigsqcup_{\mathcal{A}}\{\mathcal{F}(x,y); x$ ranges over elements of $\mathcal{B}, y$ ranges over elements of $\mathcal{C} : \mathcal{P}[x] \wedge \mathcal{Q}[y]\}$

for all values of the parameters.

The scheme *FraenkelF'R'* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, two binary functors $\mathcal{F}$ and $\mathcal{G}$ yielding sets, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(u_1,v_1); u_1$ ranges over elements of $\mathcal{A}, v_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_1,v_1]\} = \{\mathcal{G}(u_2,v_2); u_2$ ranges over elements of $\mathcal{A}, v_2$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_2,v_2]\}$

provided the parameters satisfy the following condition:

- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ such that $\mathcal{P}[u,v]$ holds $\mathcal{F}(u,v) = \mathcal{G}(u,v)$.

The scheme *FraenkelF6"R* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, two binary functors $\mathcal{F}$ and $\mathcal{G}$ yielding sets, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(u_1,v_1); u_1$ ranges over elements of $\mathcal{A}, v_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_1,v_1]\} = \{\mathcal{G}(u_2,v_2); u_2$ ranges over elements of $\mathcal{A}, v_2$ ranges over elements of $\mathcal{B} : \mathcal{Q}[u_2,v_2]\}$

provided the following requirements are met:

- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{P}[u,v]$ iff $\mathcal{Q}[u,v]$, and
- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ such that $\mathcal{P}[u,v]$ holds $\mathcal{F}(u,v) = \mathcal{G}(u,v)$.

The scheme *SupCommutativity* deals with a complete lattice $\mathcal{A}$, non empty sets $\mathcal{B}$, $\mathcal{C}$, two binary functors $\mathcal{F}$ and $\mathcal{G}$ yielding elements of $\mathcal{A}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}}\{\mathcal{F}(x,y); y$ ranges over elements of $\mathcal{C} : \mathcal{Q}[y]\}; x$ ranges over elements of $\mathcal{B} : \mathcal{P}[x]\} = \bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}}\{\mathcal{G}(x',y'); x'$ ranges over elements of $\mathcal{B} : \mathcal{P}[x']\}; y'$ ranges over elements of $\mathcal{C} : \mathcal{Q}[y']\}$

provided the parameters meet the following requirement:

- For every element x of $\mathcal{B}$ and for every element y of $\mathcal{C}$ such that $\mathcal{P}[x]$ and $\mathcal{Q}[y]$ holds $\mathcal{F}(x,y) = \mathcal{G}(x,y)$.

Next we state two propositions:

(22) Let X, Y, Z be non empty sets, R be a membership function of X, Y, S be a membership function of Y, Z, x be an element of X, and z be an element of Z. Then $(R\,S)(\langle x, z\rangle) = \bigsqcup_{\mathrm{RealPoset}[0,1]}\{R(\langle x, y\rangle) \sqcap S(\langle y, z\rangle) : y$ ranges over elements of $Y\}$.

(23) Let X, Y, Z, W be non empty sets, R be a membership function of X, Y, S be a membership function of Y, Z, and T be a membership function of Z, W. Then $(R\,S)\,T = R\,(S\,T)$.

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