JOURNAL OF FORMALIZED MATHEMATICS
Volume 12, Released 2000, Published 2003
Inst. of Computer Science, Univ. of Białystok

# Representation Theorem for Finite Distributive Lattices

Marek Dudzicz
University of Białystok

**Summary.** In the article the representation theorem for finite distributive lattice as rings of sets is presented. Auxiliary concepts are introduced. Namely, the concept of the height of an element, the maximal element in a chain, immediate predecessor of an element and ring of sets. Besides the scheme of induction in finite lattice is proved.

MML Identifier: LATTICE7.

WWW: http://mizar.org/JFM/Vol12/lattice7.html

The articles [11], [8], [15], [16], [6], [7], [13], [2], [4], [10], [18], [9], [14], [5], [1], [17], [12], and [3] provide the notation and terminology for this paper.

## 1. Induction in a Finite Lattice

Let $L$ be a 1-sorted structure and let $A$, $B$ be subsets of $L$. Let us observe that $A \subseteq B$ if and only if:

(Def. 1) For every element $x$ of $L$ such that $x \in A$ holds $x \in B$.

Let $L$ be a lattice. Observe that there exists a chain of $L$ which is non empty.

Let $L$ be a lattice and let $x$, $y$ be elements of $L$. Let us assume that $x \leq y$. A non empty chain of $L$ is called a $x$-chain of $y$ if:

(Def. 2) $x \in$ it and $y \in$ it and for every element $z$ of $L$ such that $z \in$ it holds $x \leq z$ and $z \leq y$.

We now state the proposition

(1) For every lattice $L$ and for all elements $x$, $y$ of $L$ such that $x \leq y$ holds $\{x, y\}$ is a $x$-chain of $y$.

Let $L$ be a finite lattice and let $x$ be an element of $L$. The functor height$x$ yields a natural number and is defined by:

(Def. 3) There exists a $\perp_L$-chain $A$ of $x$ such that height$x = \text{card} A$ and for every $\perp_L$-chain $A$ of $x$ holds card$A \leq$ height$x$.

One can prove the following propositions:

(2) For every finite lattice $L$ and for all elements $a$, $b$ of $L$ such that $a < b$ holds height$a <$ height$b$.

(3) Let $L$ be a finite lattice, $C$ be a chain of $L$, and $x$, $y$ be elements of $L$. If $x \in C$ and $y \in C$, then $x < y$ iff height$x <$ height$y$.

(4) Let $L$ be a finite lattice, $C$ be a chain of $L$, and $x$, $y$ be elements of $L$. If $x \in C$ and $y \in C$, then $x = y$ iff $\mathrm{height}\, x = \mathrm{height}\, y$.

(5) Let $L$ be a finite lattice, $C$ be a chain of $L$, and $x$, $y$ be elements of $L$. If $x \in C$ and $y \in C$, then $x \leq y$ iff $\mathrm{height}\, x \leq \mathrm{height}\, y$.

(6) For every finite lattice $L$ and for every element $x$ of $L$ holds $\mathrm{height}\, x = 1$ iff $x = \bot_L$.

(7) For every non empty finite lattice $L$ and for every element $x$ of $L$ holds $\mathrm{height}\, x \geq 1$.

The scheme *LattInd* deals with a finite lattice $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every element $x$ of $\mathcal{A}$ holds $\mathcal{P}[x]$

provided the parameters meet the following condition:

- For every element $x$ of $\mathcal{A}$ such that for every element $b$ of $\mathcal{A}$ such that $b < x$ holds $\mathcal{P}[b]$ holds $\mathcal{P}[x]$.

## 2. Join Irreducible Elements in a Finite Distributive Lattice

Let us observe that there exists a lattice which is distributive and finite.

Let $L$ be a lattice and let $x$, $y$ be elements of $L$. The predicate $x <_1 y$ is defined as follows:

(Def. 4) $x < y$ and it is not true that there exists an element $z$ of $L$ such that $x < z$ and $z < y$.

The following proposition is true

(8) Let $L$ be a finite lattice and $X$ be a non empty subset of $L$. Then there exists an element $x$ of $L$ such that $x \in X$ and for every element $y$ of $L$ such that $y \in X$ holds $x \not< y$.

Let $L$ be a finite lattice and let $A$ be a non empty chain of $L$. The functor $\max A$ yielding an element of $L$ is defined as follows:

(Def. 5) For every element $x$ of $L$ such that $x \in A$ holds $x \leq \max A$ and $\max A \in A$.

We now state the proposition

(9) For every finite lattice $L$ and for every element $y$ of $L$ such that $y \neq \bot_L$ there exists an element $x$ of $L$ such that $x <_1 y$.

Let $L$ be a lattice. The functor Join-IRR $L$ yielding a subset of $L$ is defined as follows:

(Def. 6) $\text{Join-IRR}\, L = \{a; a$ ranges over elements of $L$: $a \neq \bot_L \wedge \bigwedge_{b,c:\text{element of } L} (a = b \sqcup c \Rightarrow a = b \vee a = c)\}$.

Next we state three propositions:

(10) Let $L$ be a lattice and $x$ be an element of $L$. Then $x \in \text{Join-IRR}\, L$ if and only if the following conditions are satisfied:

(i) $x \neq \bot_L$, and

(ii) for all elements $b$, $c$ of $L$ such that $x = b \sqcup c$ holds $x = b$ or $x = c$.

(11) Let $L$ be a finite distributive lattice and $x$ be an element of $L$. Suppose $x \in \text{Join-IRR}\, L$. Then there exists an element $z$ of $L$ such that $z < x$ and for every element $y$ of $L$ such that $y < x$ holds $y \leq z$.

(12) For every distributive finite lattice $L$ and for every element $x$ of $L$ holds $\sup(\downarrow x \cap \text{Join-IRR}\, L) = x$.

## 3. Representation Theorem

Let $P$ be a relational structure. The functor LOWER $P$ yields a non empty set and is defined by:

(Def. 7) LOWER $P = \{X; X$ ranges over subsets of $P$: $X$ is lower$\}$.

One can prove the following two propositions:

(13) Let $L$ be a distributive finite lattice. Then there exists a map $r$ from $L$ into $\langle \mathrm{LOWER}\,\mathrm{sub}(\text{Join-IRR}\,L), \subseteq \rangle$ such that $r$ is isomorphic and for every element $a$ of $L$ holds $r(a) = \downarrow a \cap \text{Join-IRR}\,L$.

(14) For every distributive finite lattice $L$ holds $L$ and $\langle \mathrm{LOWER}\,\mathrm{sub}(\text{Join-IRR}\,L), \subseteq \rangle$ are isomorphic.

Ring of sets is defined by:

(Def. 8) It includes lattice of it.

Let us note that there exists a ring of sets which is non empty.
Let $X$ be a non empty ring of sets. Observe that $\langle X, \subseteq \rangle$ is distributive and has l.u.b.'s and g.l.b.'s.
We now state two propositions:

(15) For every finite lattice $L$ holds LOWER $\mathrm{sub}(\text{Join-IRR}\,L)$ is a ring of sets.

(16) Let $L$ be a finite lattice. Then $L$ is distributive if and only if there exists a non empty ring of sets $X$ such that $L$ and $\langle X, \subseteq \rangle$ are isomorphic.

## References


[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. Complete lattices. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/lattice3.html`.

[3] Grzegorz Bancerek. Continuous, stable, and linear maps of coherence spaces. *Journal of Formalized Mathematics*, 7, 1995. `http://mizar.org/JFM/Vol7/cohsp_1.html`.

[4] Grzegorz Bancerek. Bounds in posets and relational substructures. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/yellow_0.html`.

[5] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/waybel_0.html`.

[6] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[7] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[8] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[9] Czesław Byliński. Galois connections. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/waybel_1.html`.

[10] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/yellow_1.html`.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[12] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/pboole.html`.

[13] Wojciech A. Trybulec. Partially ordered sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/orders_1.html`.

[14] Wojciech A. Trybulec. Groups. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/group_1.html`.

[15] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[16] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[17] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_2.html`.

[18] Mariusz Żynel and Czesław Byliński. Properties of relational structures, posets, lattices and maps. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/yellow_2.html.

*Received January 6, 2000*

*Published January 2, 2004*