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Complete Lattices

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Summary. In the first section the lattice of subsets of distinct set is introduced. The join and meet operations are, respectively, union and intersection of sets, and the ordering relation is inclusion. It is shown that this lattice is Boolean, i.e. distributive and complementary. The second section introduces the poset generated in a distinct lattice by its ordering relation. Besides, it is proved that posets which have l.u.b.'s and g.l.b.'s for every two elements generate lattices with the same ordering relations. In the last section the concept of complete lattice is introduced and discussed. Finally, the fact that the function f from subsets of distinct set yielding elements of this set is a infinite union of some complete lattice, if f yields an element a for singleton $\{a\}$ and $f(f^{\circ}X) = f(\bigsqcup X)$ for every subset X, is proved. Some concepts and proofs are based on [8] and [9].

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The articles [11], [7], [14], [10], [4], [5], [3], [18], [1], [12], [2], [15], [17], [16], [6], and [13] provide the notation and terminology for this paper.

1. Boolean Lattice of Subsets

Let X be a set. The lattice of subsets of X yields a strict lattice structure and is defined by the conditions (Def. 1).

(Def. 1)(i) The carrier of the lattice of subsets of $X = 2^X$, and

(ii) for all elements Y, Z of 2^X holds (the join operation of the lattice of subsets of $X)(Y, Z) = Y \cup Z$ and (the meet operation of the lattice of subsets of $X)(Y, Z) = Y \cap Z$.

In the sequel X is a set and x, y are elements of the lattice of subsets of X.

Let X be a set. One can check that the lattice of subsets of X is non empty.

One can prove the following two propositions:

(1) $x \sqcup y = x \cup y$ and $x \sqcap y = x \cap y$.

(2) $x \sqsubseteq y$ iff $x \subseteq y$.

Let us consider X. Observe that the lattice of subsets of X is lattice-like.

The following propositions are true:

(3) The lattice of subsets of X is lower-bounded and $\perp_{\text{the lattice of subsets of } X} = \emptyset$.

(4) The lattice of subsets of X is upper-bounded and $\top_{\text{the lattice of subsets of } X} = X$.

Let us consider X. Observe that the lattice of subsets of X is Boolean and lattice-like.

One can prove the following proposition

(5) For every element x of the lattice of subsets of X holds $x^{\rm c} = X \setminus x$.

2. Correspondence between Lattices and Posets

Let L be a lattice. Then $\mathrm{LattRel}(L)$ is an order in the carrier of L.

Let L be a lattice. The functor $\mathrm{Poset}(L)$ yields a strict poset and is defined by:

(Def. 2) $\mathrm{Poset}(L) = \langle$the carrier of L, $\mathrm{LattRel}(L)\rangle$.

Let L be a lattice. Note that $\mathrm{Poset}(L)$ is non empty.

Next we state the proposition

(6) For all lattices L_1, L_2 such that $\mathrm{Poset}(L_1) = \mathrm{Poset}(L_2)$ holds the lattice structure of $L_1 =$ the lattice structure of L_2.

Let L be a lattice and let p be an element of L. The functor $p^{\bullet}$ yields an element of $\mathrm{Poset}(L)$ and is defined as follows:

(Def. 3) $p^{\bullet} = p$.

Let L be a lattice and let p be an element of $\mathrm{Poset}(L)$. The functor ${}^{\bullet}p$ yielding an element of L is defined by:

(Def. 4) ${}^{\bullet}p = p$.

In the sequel L is a lattice and p, q are elements of L.

We now state the proposition

(7) $p \sqsubseteq q$ iff $p^{\bullet} \leq q^{\bullet}$.

Let X be a set and let O be an order in X. Then $O^{\smile}$ is an order in X.

Let A be a relational structure. The functor $A^{\smile}$ yields a strict relational structure and is defined as follows:

(Def. 5) $A^{\smile} = \langle$the carrier of A, (the internal relation of $A)^{\smile}\rangle$.

Let A be a poset. Observe that $A^{\smile}$ is reflexive, transitive, and antisymmetric.

Let A be a non empty relational structure. Note that $A^{\smile}$ is non empty.

In the sequel A denotes a relational structure and a, b denote elements of A.

Next we state the proposition

(8) $(A^{\smile})^{\smile} =$ the relational structure of A.

Let A be a relational structure and let a be an element of A. The functor $a^{\smile}$ yielding an element of $A^{\smile}$ is defined by:

(Def. 6) $a^{\smile} = a$.

Let A be a relational structure and let a be an element of $A^{\smile}$. The functor ${\frown}a$ yields an element of A and is defined as follows:

(Def. 7) ${\frown}a = a$.

Next we state the proposition

(9) $a \leq b$ iff $b^{\smile} \leq a^{\smile}$.

Let A be a relational structure, let X be a set, and let a be an element of A. The predicate $a \leq X$ is defined by:

(Def. 8) For every element b of A such that $b \in X$ holds $a \leq b$.

We introduce $X \geq a$ as a synonym of $a \leq X$. The predicate $X \leq a$ is defined as follows:

(Def. 9) For every element b of A such that $b \in X$ holds $b \leq a$.

We introduce $a \geq X$ as a synonym of $X \leq a$.

Let I_1 be a relational structure. We say that I_1 has l.u.b.'s if and only if the condition (Def. 10) is satisfied.

(Def. 10) Let x, y be elements of I_1. Then there exists an element z of I_1 such that $x \leq z$ and $y \leq z$ and for every element z' of I_1 such that $x \leq z'$ and $y \leq z'$ holds $z \leq z'$.

We say that I_1 has g.l.b.'s if and only if the condition (Def. 11) is satisfied.

(Def. 11) Let x, y be elements of I_1. Then there exists an element z of I_1 such that $z \leq x$ and $z \leq y$ and for every element z' of I_1 such that $z' \leq x$ and $z' \leq y$ holds $z' \leq z$.

Let us note that every relational structure which has l.u.b.'s is also non empty and every relational structure which has g.l.b.'s is also non empty.

One can prove the following propositions:

(10) A has l.u.b.'s iff $A^{\smile}$ has g.l.b.'s.

(11) For every lattice L holds $\mathrm{Poset}(L)$ has l.u.b.'s and g.l.b.'s.

Let I_1 be a relational structure. We say that I_1 is complete if and only if:

(Def. 12) For every set X there exists an element a of I_1 such that $X \leq a$ and for every element b of I_1 such that $X \leq b$ holds $a \leq b$.

Let us note that there exists a poset which is strict, complete, and non empty.

In the sequel A denotes a non empty relational structure.

The following proposition is true

(12) If A is complete, then A has l.u.b.'s and g.l.b.'s.

Let us note that there exists a poset which is complete, strict, and non empty and has l.u.b.'s and g.l.b.'s.

Let A be a relational structure. Let us assume that A is antisymmetric. Let a, b be elements of A. Let us assume that there exists an element x of A such that $a \leq x$ and $b \leq x$ and for every element c of A such that $a \leq c$ and $b \leq c$ holds $x \leq c$. The functor $a \sqcup b$ yielding an element of A is defined as follows:

(Def. 13) $a \leq a \sqcup b$ and $b \leq a \sqcup b$ and for every element c of A such that $a \leq c$ and $b \leq c$ holds $a \sqcup b \leq c$.

Let A be a relational structure. Let us assume that A is antisymmetric. Let a, b be elements of A. Let us assume that there exists an element x of A such that $a \geq x$ and $b \geq x$ and for every element c of A such that $a \geq c$ and $b \geq c$ holds $x \geq c$. The functor $a \sqcap b$ yields an element of A and is defined as follows:

(Def. 14) $a \sqcap b \leq a$ and $a \sqcap b \leq b$ and for every element c of A such that $c \leq a$ and $c \leq b$ holds $c \leq a \sqcap b$.

For simplicity, we use the following convention: V denotes an antisymmetric relational structure with l.u.b.'s, u_1, u_2, u_3 denote elements of V, N denotes an antisymmetric relational structure with g.l.b.'s, n_1, n_2, n_3 denote elements of N, K denotes a reflexive antisymmetric relational structure with l.u.b.'s and g.l.b.'s, and k_1, k_2 denote elements of K.

One can prove the following propositions:

(13) $u_1 \sqcup u_2 = u_2 \sqcup u_1$.

(14) If V is transitive, then $(u_1 \sqcup u_2) \sqcup u_3 = u_1 \sqcup (u_2 \sqcup u_3)$.

(15) $n_1 \sqcap n_2 = n_2 \sqcap n_1$.

(16) If N is transitive, then $(n_1 \sqcap n_2) \sqcap n_3 = n_1 \sqcap (n_2 \sqcap n_3)$.

Let L be an antisymmetric relational structure with g.l.b.'s and let x, y be elements of L. Let us observe that the functor $x \sqcap y$ is commutative.

Let L be an antisymmetric relational structure with l.u.b.'s and let x, y be elements of L. Let us observe that the functor $x \sqcup y$ is commutative.

Next we state three propositions:

(17) $(k_1 \sqcap k_2) \sqcup k_2 = k_2$.

(18) $k_1 \sqcap (k_1 \sqcup k_2) = k_1$.

(19) For every poset A with l.u.b.'s and g.l.b.'s there exists a strict lattice L such that the relational structure of $A = \mathrm{Poset}(L)$.

Let A be a relational structure. Let us assume that A is a poset with l.u.b.'s and g.l.b.'s. The functor $\mathbb{L}_A$ yielding a strict lattice is defined by:

(Def. 15) The relational structure of $A = \mathrm{Poset}(\mathbb{L}_A)$.

The following proposition is true

(20) $\mathrm{LattRel}(L)^{\smile} = \mathrm{LattRel}(L^{\circ})$ and $\mathrm{Poset}(L)^{\smile} = \mathrm{Poset}(L^{\circ})$.

3. Complete lattices

Let L be a non empty lattice structure, let p be an element of L, and let X be a set. The predicate $p \sqsubseteq X$ is defined as follows:

(Def. 16) For every element q of L such that $q \in X$ holds $p \sqsubseteq q$.

We introduce $X \sqsupseteq p$ as a synonym of $p \sqsubseteq X$. The predicate $X \sqsubseteq p$ is defined as follows:

(Def. 17) For every element q of L such that $q \in X$ holds $q \sqsubseteq p$.

We introduce $p \sqsupseteq X$ as a synonym of $X \sqsubseteq p$.

We now state two propositions:

(21) For every lattice L and for all elements p, q, r of L holds $p \sqsubseteq \{q, r\}$ iff $p \sqsubseteq q \sqcap r$.

(22) For every lattice L and for all elements p, q, r of L holds $p \sqsupseteq \{q, r\}$ iff $q \sqcup r \sqsubseteq p$.

Let I_1 be a non empty lattice structure. We say that I_1 is complete if and only if:

(Def. 18) For every set X there exists an element p of I_1 such that $X \sqsubseteq p$ and for every element r of I_1 such that $X \sqsubseteq r$ holds $p \sqsubseteq r$.

We say that I_1 is $\bigsqcup$-distributive if and only if the condition (Def. 19) is satisfied.

(Def. 19) Let given X and a, b, c be elements of I_1. Suppose that

(i) $X \sqsubseteq a$,

(ii) for every element d of I_1 such that $X \sqsubseteq d$ holds $a \sqsubseteq d$,

(iii) $\{b \sqcap a'; a'$ ranges over elements of I_1: $a' \in X\} \sqsubseteq c$, and

(iv) for every element d of I_1 such that $\{b \sqcap a'; a'$ ranges over elements of I_1: $a' \in X\} \sqsubseteq d$ holds $c \sqsubseteq d$.

Then $b \sqcap a \sqsubseteq c$.

We say that I_1 is $\bigsqcap$-distributive if and only if the condition (Def. 20) is satisfied.

(Def. 20) Let given X and a, b, c be elements of I_1. Suppose that

(i) $X \sqsupseteq a$,

(ii) for every element d of I_1 such that $X \sqsupseteq d$ holds $d \sqsubseteq a$,

(iii) $\{b \sqcup a'; a'$ ranges over elements of I_1: $a' \in X\} \sqsupseteq c$, and

(iv) for every element d of I_1 such that $\{b \sqcup a'; a'$ ranges over elements of I_1: $a' \in X\} \sqsupseteq d$ holds $d \sqsubseteq c$.

Then $c \sqsubseteq b \sqcup a$.

We now state several propositions:

(23) Let B be a Boolean lattice and a be an element of B. Then $X \sqsubseteq a$ if and only if $\{b^{\mathrm{c}}; b$ ranges over elements of B: $b \in X\} \sqsupseteq a^{\mathrm{c}}$.

(24) Let B be a Boolean lattice and a be an element of B. Then $X \sqsupseteq a$ if and only if $\{b^{\mathrm{c}}; b$ ranges over elements of B: $b \in X\} \sqsubseteq a^{\mathrm{c}}$.

(25) The lattice of subsets of X is complete.

(26) The lattice of subsets of X is $\bigsqcup$-distributive.

(27) The lattice of subsets of X is $\bigsqcap$-distributive.

One can verify that there exists a lattice which is complete, $\bigsqcup$-distributive, $\bigsqcap$-distributive, and strict.

In the sequel p' is an element of $\mathrm{Poset}(L)$.

One can prove the following propositions:

(28) $p \sqsubseteq X$ iff $p^{\bullet} \leq X$.

(29) $p' \leq X$ iff ${}^{\bullet}p' \sqsubseteq X$.

(30) $X \sqsubseteq p$ iff $X \leq p^{\bullet}$.

(31) $X \leq p'$ iff $X \sqsubseteq {}^{\bullet}p'$.

Let A be a complete non empty poset. Observe that $\mathbb{L}_A$ is complete.

Let L be a non empty lattice structure. Let us assume that L is a complete lattice. Let X be a set. The functor $\bigsqcup_L X$ yields an element of L and is defined by:

(Def. 21) $X \sqsubseteq \bigsqcup_L X$ and for every element r of L such that $X \sqsubseteq r$ holds $\bigsqcup_L X \sqsubseteq r$.

Let L be a non empty lattice structure and let X be a set. The functor $\bigsqcap_L X$ yielding an element of L is defined as follows:

(Def. 22) $\bigsqcap_L X = \bigsqcup_L \{p; p$ ranges over elements of L: $p \sqsubseteq X\}$.

Let L be a non empty lattice structure and let X be a subset of L. We introduce $\bigsqcup X$ as a synonym of $\bigsqcup_L X$. We introduce $\bigsqcap X$ as a synonym of $\bigsqcap_L X$.

We use the following convention: C is a complete lattice, a, b, c are elements of C, and X, Y are sets.

Next we state a number of propositions:

(32) $\bigsqcup_C \{a \sqcap b : b \in X\} \sqsubseteq a \sqcap \bigsqcup_C X$.

(33) C is $\bigsqcup$-distributive iff for all X, a holds $a \sqcap \bigsqcup_C X \sqsubseteq \bigsqcup_C \{a \sqcap b : b \in X\}$.

(34) $a = \bigsqcap_C X$ iff $a \sqsubseteq X$ and for every b such that $b \sqsubseteq X$ holds $b \sqsubseteq a$.

(35) $a \sqcup \bigsqcap_C X \sqsubseteq \bigsqcap_C \{a \sqcup b : b \in X\}$.

(36) C is $\bigsqcap$-distributive iff for all X, a holds $\bigsqcap_C\{a \sqcup b : b \in X\} \sqsubseteq a \sqcup \bigsqcap_C X$.

(37) $\bigsqcup_C X = \bigsqcap_C\{a : a \sqsupseteq X\}$.

(38) If $a \in X$, then $a \sqsubseteq \bigsqcup_C X$ and $\bigsqcap_C X \sqsubseteq a$.

(40)[1] If $a \sqsubseteq X$, then $a \sqsubseteq \bigsqcap_C X$.

(41) If $a \in X$ and $X \sqsubseteq a$, then $\bigsqcup_C X = a$.

(42) If $a \in X$ and $a \sqsubseteq X$, then $\bigsqcap_C X = a$.

(43) $\bigsqcup\{a\} = a$ and $\bigsqcap\{a\} = a$.

(44) $a \sqcup b = \bigsqcup\{a,b\}$ and $a \sqcap b = \bigsqcap\{a,b\}$.

(45) $a = \bigsqcup_C\{b : b \sqsubseteq a\}$ and $a = \bigsqcap_C\{c : a \sqsubseteq c\}$.

(46) If $X \subseteq Y$, then $\bigsqcup_C X \sqsubseteq \bigsqcup_C Y$ and $\bigsqcap_C Y \sqsubseteq \bigsqcap_C X$.

(47) $\bigsqcup_C X = \bigsqcup_C\{a : \bigvee_b \ (a \sqsubseteq b \ \wedge \ b \in X)\}$ and $\bigsqcap_C X = \bigsqcap_C\{b : \bigvee_a \ (a \sqsubseteq b \ \wedge \ a \in X)\}$.

(48) If for every a such that $a \in X$ there exists b such that $a \sqsubseteq b$ and $b \in Y$, then $\bigsqcup_C X \sqsubseteq \bigsqcup_C Y$.

(49) If $X \subseteq 2^{\text{the carrier of } C}$, then $\bigsqcup_C \bigcup X = \bigsqcup_C\{\bigsqcup Y; Y$ ranges over subsets of C: $Y \in X\}$.

(50) C is a lower bound lattice and $\perp_C = \bigsqcup_C \emptyset$.

(51) C is an upper bound lattice and $\top_C = \bigsqcup_C$ (the carrier of C).

(52) If C is an implicative lattice, then $a \Rightarrow b = \bigsqcup_C\{c : a \sqcap c \sqsubseteq b\}$.

(53) If C is an implicative lattice, then C is $\bigsqcup$-distributive.

(54) Let D be a complete $\bigsqcup$-distributive lattice and a be an element of D. Then $a \sqcap \bigsqcup_D X = \bigsqcup_D\{a \sqcap b_1; b_1$ ranges over elements of D: $b_1 \in X\}$ and $\bigsqcup_D X \sqcap a = \bigsqcup_D\{b_2 \sqcap a; b_2$ ranges over elements of D: $b_2 \in X\}$.

(55) Let D be a complete $\bigsqcap$-distributive lattice and a be an element of D. Then $a \sqcup \bigsqcap_D X = \bigsqcap_D\{a \sqcup b_1; b_1$ ranges over elements of D: $b_1 \in X\}$ and $\bigsqcap_D X \sqcup a = \bigsqcap_D\{b_2 \sqcup a; b_2$ ranges over elements of D: $b_2 \in X\}$.

In this article we present several logical schemes. The scheme *SingleFraenkel* deals with a set $\mathcal{A}$, a non empty set $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{A}; a$ ranges over elements of $\mathcal{B} : \mathcal{P}[a]\} = \{\mathcal{A}\}$

provided the following condition is met:

- There exists an element a of $\mathcal{B}$ such that $\mathcal{P}[a]$.

The scheme *FuncFraenkel* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a function $\mathcal{C}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{C}^\circ\{\mathcal{F}(x); x$ ranges over elements of $\mathcal{A} : \mathcal{P}[x]\} = \{\mathcal{C}(\mathcal{F}(x)); x$ ranges over elements of $\mathcal{A} : \mathcal{P}[x]\}$

provided the following requirement is met:

- $\mathcal{B} \subseteq \operatorname{dom} \mathcal{C}$.

Next we state the proposition

(56) Let D be a non empty set and f be a function from 2^D into D. Suppose for every element a of D holds $f(\{a\}) = a$ and for every subset X of 2^D holds $f(f^\circ X) = f(\bigcup X)$. Then there exists a complete strict lattice L such that the carrier of $L = D$ and for every subset X of L holds $\bigsqcup X = f(X)$.

[1] The proposition (39) has been removed.

REFERENCES

[1] Grzegorz Bancerek. Filters — part I. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/filter_0.html`.

[2] Grzegorz Bancerek. Filters - part II. Quotient lattices modulo filters and direct product of two lattices. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/filter_1.html`.

[3] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/binop_1.html`.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[6] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[7] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[8] Helena Rasiowa and Roman Sikorski. *The Mathematics of Metamathematics*, volume 41 of *Monografie Matematyczne*. PWN, Warszawa, 1968.

[9] Tadeusz Traczyk. *Wstęp do teorii algebr Boole'a*, volume 37 of *Biblioteka Matematyczna*. PWN, Warszawa, 1970.

[10] Andrzej Trybulec. Domains and their Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/domain_1.html`.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[12] Andrzej Trybulec. Finite join and finite meet, and dual lattices. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/lattice2.html`.

[13] Wojciech A. Trybulec. Partially ordered sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/orders_1.html`.

[14] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[15] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[16] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

[17] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_2.html`.

[18] Stanisław Żukowski. Introduction to lattice theory. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/lattices.html`.

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