JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Finite Join and Finite Meet, and Dual Lattices

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Summary. The concepts of finite join and finite meet in a lattice are introduced. Some properties of the finite join are proved. After introducing the concept of dual lattice in view of dualism we obtain analogous properties of the meet. We prove these properties of binary operations in a lattice, which are usually included in axioms of the lattice theory. We also introduce the concept of Heyting lattice (a bounded lattice with relative pseudo-complements).

MML Identifier: `LATTICE2`.

WWW: `http://mizar.org/JFM/Vol2/lattice2.html`

The articles [10], [13], [14], [3], [4], [6], [5], [8], [2], [15], [7], [11], [12], [9], and [1] provide the notation and terminology for this paper.

For simplicity, we follow the rules: A is a set, C is a non empty set, B is a subset of A, x is an element of A, and f, g are functions from A into C.

One can prove the following propositions:

(2)[1] $\mathrm{dom}(g \restriction B) = B$.

(5)[2] $f \restriction B = g \restriction B$ iff for every x such that $x \in B$ holds $g(x) = f(x)$.

(6) For every set B holds $f {+\cdot} g \restriction B$ is a function from A into C.

(7) $g \restriction B {+\cdot} f = f$.

(8) For all functions f, g such that $g \leq f$ holds $f {+\cdot} g = f$.

(9) $f {+\cdot} f \restriction B = f$.

(10) If for every x such that $x \in B$ holds $g(x) = f(x)$, then $f {+\cdot} g \restriction B = f$.

In the sequel B denotes a finite subset of A.

Next we state four propositions:

(12)[3] $g \restriction B {+\cdot} f = f$.

(13) $\mathrm{dom}(g \restriction B) = B$.

(14) If for every x such that $x \in B$ holds $g(x) = f(x)$, then $f {+\cdot} g \restriction B = f$.

(16)[4] If $f \restriction B = g \restriction B$, then $f^\circ B = g^\circ B$.

[1] The proposition (1) has been removed.
[2] The propositions (3) and (4) have been removed.
[3] The proposition (11) has been removed.
[4] The proposition (15) has been removed.

Let D be a non empty set and let o, o' be binary operations on D. We say that o absorbs o' if and only if:

(Def. 1) For all elements x, y of D holds $o(x, o'(x, y)) = x$.

We introduce o does not absorb o' as an antonym of o absorbs o'.

In the sequel L denotes a non empty lattice structure.

One can prove the following proposition

(17) Suppose that

(i) the join operation of L is commutative and associative,

(ii) the meet operation of L is commutative and associative,

(iii) the join operation of L absorbs the meet operation of L, and

(iv) the meet operation of L absorbs the join operation of L.

Then L is lattice-like.

Let L be a lattice structure. The functor L° yields a strict lattice structure and is defined as follows:

(Def. 2) $L^\circ = \langle$the carrier of L, the meet operation of L, the join operation of $L\rangle$.

Let L be a non empty lattice structure. Observe that L° is non empty.

We now state two propositions:

(18)(i) The carrier of $L =$ the carrier of L°,

(ii) the join operation of $L =$ the meet operation of L°, and

(iii) the meet operation of $L =$ the join operation of L°.

(19) For every strict non empty lattice structure L holds $(L^\circ)^\circ = L$.

We adopt the following rules: L denotes a lattice and a, b, u, v denote elements of L.

The following propositions are true:

(21)[5] If for every v holds $u \sqcup v = v$, then $u = \bot_L$.

(22) If for every v holds (the join operation of $L)(u, v) = v$, then $u = \bot_L$.

(24)[6] If for every v holds $u \sqcap v = v$, then $u = \top_L$.

(25) If for every v holds (the meet operation of $L)(u, v) = v$, then $u = \top_L$.

(26) The join operation of L is idempotent.

(27) Let L be a join-commutative non empty $\sqcup$-semi lattice structure. Then the join operation of L is commutative.

(28) If the join operation of L has a unity, then $\bot_L = \mathbf{1}_{\text{the join operation of } L}$.

(29) Let L be a join-associative non empty $\sqcup$-semi lattice structure. Then the join operation of L is associative.

(30) The meet operation of L is idempotent.

(31) Let L be a meet-commutative non empty $\sqcap$-semi lattice structure. Then the meet operation of L is commutative.

(32) Let L be a meet-associative non empty $\sqcap$-semi lattice structure. Then the meet operation of L is associative.

[5] The proposition (20) has been removed.

[6] The proposition (23) has been removed.

Let L be a join-commutative non empty $\sqcup$-semi lattice structure. Note that the join operation of L is commutative.

Let L be a join-associative non empty $\sqcup$-semi lattice structure. One can verify that the join operation of L is associative.

Let L be a meet-commutative non empty $\sqcap$-semi lattice structure. Observe that the meet operation of L is commutative.

Let L be a meet-associative non empty $\sqcap$-semi lattice structure. Note that the meet operation of L is associative.

The following propositions are true:

(33) If the meet operation of L has a unity, then $\top_L = \mathbf{1}_{\text{the meet operation of } L}$.

(34) The join operation of L is distributive w.r.t. the join operation of L.

(35) Suppose L is a distributive lattice. Then the join operation of L is distributive w.r.t. the meet operation of L.

(36) If the join operation of L is distributive w.r.t. the meet operation of L, then L is distributive.

(37) Suppose L is a distributive lattice. Then the meet operation of L is distributive w.r.t. the join operation of L.

(38) If the meet operation of L is distributive w.r.t. the join operation of L, then L is distributive.

(39) The meet operation of L is distributive w.r.t. the meet operation of L.

(40) The join operation of L absorbs the meet operation of L.

(41) The meet operation of L absorbs the join operation of L.

Let A be a non empty set, let L be a lattice, let B be a finite subset of A, and let f be a function from A into the carrier of L. The functor $\bigsqcup^{\rm f}_B f$ yielding an element of L is defined as follows:

(Def. 3) $\bigsqcup^{\rm f}_B f =$ (the join operation of L)-$\sum_B f$.

The functor $\bigsqcap^{\rm f}_B f$ yields an element of L and is defined as follows:

(Def. 4) $\bigsqcap^{\rm f}_B f =$ (the meet operation of L)-$\sum_B f$.

For simplicity, we adopt the following rules: A denotes a non empty set, x denotes an element of A, B denotes a finite subset of A, and f, g denote functions from A into the carrier of L.

One can prove the following propositions:

(43)[7] If $x \in B$, then $f(x) \sqsubseteq \bigsqcup^{\rm f}_B f$.

(44) If there exists x such that $x \in B$ and $u \sqsubseteq f(x)$, then $u \sqsubseteq \bigsqcup^{\rm f}_B f$.

(45) If for every x such that $x \in B$ holds $f(x) = u$ and $B \neq \emptyset$, then $\bigsqcup^{\rm f}_B f = u$.

(46) If $\bigsqcup^{\rm f}_B f \sqsubseteq u$, then for every x such that $x \in B$ holds $f(x) \sqsubseteq u$.

(47) If $B \neq \emptyset$ and for every x such that $x \in B$ holds $f(x) \sqsubseteq u$, then $\bigsqcup^{\rm f}_B f \sqsubseteq u$.

(48) If $B \neq \emptyset$ and for every x such that $x \in B$ holds $f(x) \sqsubseteq g(x)$, then $\bigsqcup^{\rm f}_B f \sqsubseteq \bigsqcup^{\rm f}_B g$.

(49) If $B \neq \emptyset$ and $f \restriction B = g \restriction B$, then $\bigsqcup^{\rm f}_B f = \bigsqcup^{\rm f}_B g$.

(50) If $B \neq \emptyset$, then $v \sqcup \bigsqcup^{\rm f}_B f = \bigsqcup^{\rm f}_B(($the join operation of $L)^{\circ}(v, f))$.

Let L be a lattice. One can check that L° is lattice-like.

We now state a number of propositions:

[7] The proposition (42) has been removed.

(51) Let L be a lattice, B be a finite subset of A, f be a function from A into the carrier of L, and f' be a function from A into the carrier of L°. If $f = f'$, then $\bigsqcup^{\rm f}_{B} f = \bigsqcap^{\rm f}_{B} f'$ and $\bigsqcap^{\rm f}_{B} f = \bigsqcup^{\rm f}_{B} f'$.

(52) For all elements a', b' of L° such that $a = a'$ and $b = b'$ holds $a \sqcap b = a' \sqcup b'$ and $a \sqcup b = a' \sqcap b'$.

(53) If $a \sqsubseteq b$, then for all elements a', b' of L° such that $a = a'$ and $b = b'$ holds $b' \sqsubseteq a'$.

(54) For all elements a', b' of L° such that $a' \sqsubseteq b'$ and $a = a'$ and $b = b'$ holds $b \sqsubseteq a$.

(55) If $x \in B$, then $\bigsqcap^{\rm f}_{B} f \sqsubseteq f(x)$.

(56) If there exists x such that $x \in B$ and $f(x) \sqsubseteq u$, then $\bigsqcap^{\rm f}_{B} f \sqsubseteq u$.

(57) If for every x such that $x \in B$ holds $f(x) = u$ and $B \neq \emptyset$, then $\bigsqcap^{\rm f}_{B} f = u$.

(58) If $B \neq \emptyset$, then $v \sqcap \bigsqcap^{\rm f}_{B} f = \bigsqcap^{\rm f}_{B}((\text{the meet operation of } L)^{\circ}(v, f))$.

(59) If $u \sqsubseteq \bigsqcap^{\rm f}_{B} f$, then for every x such that $x \in B$ holds $u \sqsubseteq f(x)$.

(60) If $B \neq \emptyset$ and $f{\restriction}B = g{\restriction}B$, then $\bigsqcap^{\rm f}_{B} f = \bigsqcap^{\rm f}_{B} g$.

(61) If $B \neq \emptyset$ and for every x such that $x \in B$ holds $u \sqsubseteq f(x)$, then $u \sqsubseteq \bigsqcap^{\rm f}_{B} f$.

(62) If $B \neq \emptyset$ and for every x such that $x \in B$ holds $f(x) \sqsubseteq g(x)$, then $\bigsqcap^{\rm f}_{B} f \sqsubseteq \bigsqcap^{\rm f}_{B} g$.

(63) For every lattice L holds L is lower-bounded iff L° is upper-bounded.

(64) For every lattice L holds L is upper-bounded iff L° is lower-bounded.

(65) L is a distributive lattice iff L° is a distributive lattice.

In the sequel L is a lower bound lattice, f, g are functions from A into the carrier of L, and u is an element of L.

One can prove the following propositions:

(66) $\perp_{L}$ is a unity w.r.t. the join operation of L.

(67) The join operation of L has a unity.

(68) $\perp_{L} = \mathbf{1}_{\text{the join operation of } L}$.

(69) If $f{\restriction}B = g{\restriction}B$, then $\bigsqcup^{\rm f}_{B} f = \bigsqcup^{\rm f}_{B} g$.

(70) If for every x such that $x \in B$ holds $f(x) \sqsubseteq u$, then $\bigsqcup^{\rm f}_{B} f \sqsubseteq u$.

(71) If for every x such that $x \in B$ holds $f(x) \sqsubseteq g(x)$, then $\bigsqcup^{\rm f}_{B} f \sqsubseteq \bigsqcup^{\rm f}_{B} g$.

In the sequel L denotes an upper bound lattice, f, g denote functions from A into the carrier of L, and u denotes an element of L.

The following propositions are true:

(72) $\top_{L}$ is a unity w.r.t. the meet operation of L.

(73) The meet operation of L has a unity.

(74) $\top_{L} = \mathbf{1}_{\text{the meet operation of } L}$.

(75) If $f{\restriction}B = g{\restriction}B$, then $\bigsqcap^{\rm f}_{B} f = \bigsqcap^{\rm f}_{B} g$.

(76) If for every x such that $x \in B$ holds $u \sqsubseteq f(x)$, then $u \sqsubseteq \bigsqcap^{\rm f}_{B} f$.

(77) If for every x such that $x \in B$ holds $f(x) \sqsubseteq g(x)$, then $\bigsqcap^{\rm f}_{B} f \sqsubseteq \bigsqcap^{\rm f}_{B} g$.

(78) For every lower bound lattice L holds $\perp_{L} = \top_{L^{\circ}}$.

(79) For every upper bound lattice L holds $\top_L = \bot_{L^\circ}$.

A distributive lower bounded lattice is a distributive lower-bounded lattice.

In the sequel L denotes a distributive lower bounded lattice, f, g denote functions from A into the carrier of L, and u denotes an element of L.

One can prove the following propositions:

(80) The meet operation of L is distributive w.r.t. the join operation of L.

(81) (The meet operation of $L)(u, \bigsqcup^{\rm f}_B f) = \bigsqcup^{\rm f}_B(($the meet operation of $L)^\circ(u, f))$.

(82) If for every x such that $x \in B$ holds $g(x) = u \sqcap f(x)$, then $u \sqcap \bigsqcup^{\rm f}_B f = \bigsqcup^{\rm f}_B g$.

(83) $u \sqcap \bigsqcup^{\rm f}_B f = \bigsqcup^{\rm f}_B(($the meet operation of $L)^\circ(u, f))$.

Let I_1 be a lattice. We say that I_1 is Heyting if and only if:

(Def. 6)[8] I_1 is implicative and lower-bounded.

Let us observe that there exists a lattice which is Heyting.

One can check that every lattice which is Heyting is also implicative and lower-bounded and every lattice which is implicative and lower-bounded is also Heyting.

A Heyting lattice is a Heyting lattice.

Let us note that there exists a lattice which is Heyting and strict.

Next we state two propositions:

(84) Let L be a lower bound lattice. Then L is a Heyting lattice if and only if for all elements x, z of L there exists an element y of L such that $x \sqcap y \sqsubseteq z$ and for every element v of L such that $x \sqcap v \sqsubseteq z$ holds $v \sqsubseteq y$.

(85) For every lattice L holds L is finite iff L° is finite.

Let us note that every lattice which is finite is also lower-bounded and every lattice which is finite is also upper-bounded.

Let us observe that every lattice which is finite is also bounded.

One can check that every lattice which is distributive and finite is also Heyting.

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Received August 10, 1990

Published January 2, 2004