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The de l'Hospital Theorem

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Summary. List of theorems concerning the de l'Hospital Theorem. We discuss the case when both functions have the zero value at a point and when the quotient of their differentials is convergent at this point.

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The articles [13], [1], [14], [2], [4], [3], [15], [8], [12], [9], [10], [11], [6], [7], and [5] provide the notation and terminology for this paper.

We adopt the following convention: f, g are partial functions from $\mathbb{R}$ to $\mathbb{R}$, r, r_1, r_2, g_1, g_2, x_0, t are real numbers, and a is a sequence of real numbers.

We now state a number of propositions:

(1) Suppose that

(i) f is continuous in x_0, and

(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$.

Then f is convergent in x_0.

(2) f is right convergent in x_0 and $\lim_{x_0^+} f = t$ if and only if the following conditions are satisfied:

(i) for every r such that $x_0 < r$ there exists t such that $t < r$ and $x_0 < t$ and $t \in \operatorname{dom} f$, and

(ii) for every a such that a is convergent and $\lim a = x_0$ and $\operatorname{rng} a \subseteq \operatorname{dom} f \cap]x_0, +\infty[$ holds $f \cdot a$ is convergent and $\lim(f \cdot a) = t$.

(3) f is left convergent in x_0 and $\lim_{x_0^-} f = t$ if and only if the following conditions are satisfied:

(i) for every r such that $r < x_0$ there exists t such that $r < t$ and $t < x_0$ and $t \in \operatorname{dom} f$, and

(ii) for every a such that a is convergent and $\lim a = x_0$ and $\operatorname{rng} a \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ holds $f \cdot a$ is convergent and $\lim(f \cdot a) = t$.

(4) Given a neighbourhood N of x_0 such that $N \setminus \{x_0\} \subseteq \operatorname{dom} f$. Let given r_1, r_2. Suppose $r_1 < x_0$ and $x_0 < r_2$. Then there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$.

(5) Given a neighbourhood N of x_0 such that

f is differentiable on N and g is differentiable on N and $N \setminus \{x_0\} \subseteq \mathrm{dom}(\frac{f}{g})$ and $N \subseteq \mathrm{dom}(\frac{f'_{\restriction N}}{g'_{\restriction N}})$ and $f(x_0) = 0$ and $g(x_0) = 0$ and $\frac{f'_{\restriction N}}{g'_{\restriction N}}$ is divergent to $+\infty$ in x_0. Then $\frac{f}{g}$ is divergent to $+\infty$ in x_0.

(6) Given a neighbourhood N of x_0 such that

f is differentiable on N and g is differentiable on N and $N \setminus \{x_0\} \subseteq \mathrm{dom}(\frac{f}{g})$ and $N \subseteq \mathrm{dom}(\frac{f'_{\restriction N}}{g'_{\restriction N}})$ and $f(x_0) = 0$ and $g(x_0) = 0$ and $\frac{f'_{\restriction N}}{g'_{\restriction N}}$ is divergent to $-\infty$ in x_0. Then $\frac{f}{g}$ is divergent to $-\infty$ in x_0.

(7) Given r such that

$r > 0$ and f is differentiable on $]x_0, x_0 + r[$ and g is differentiable on $]x_0, x_0 + r[$ and $]x_0, x_0 + r[\subseteq \mathrm{dom}(\frac{f}{g})$ and $[x_0, x_0 + r] \subseteq \mathrm{dom}(\frac{f'_{\restriction]x_0, x_0+r[}}{g'_{\restriction]x_0, x_0+r[}})$ and $f(x_0) = 0$ and $g(x_0) = 0$ and f is continuous in x_0 and g is continuous in x_0 and $\frac{f'_{\restriction]x_0, x_0+r[}}{g'_{\restriction]x_0, x_0+r[}}$ is right convergent in x_0. Then $\frac{f}{g}$ is right convergent in x_0 and there exists r such that $r > 0$ and $\lim_{x_0^+}(\frac{f}{g}) = \lim_{x_0^+}(\frac{f'_{\restriction]x_0, x_0+r[}}{g'_{\restriction]x_0, x_0+r[}})$.

(8) Given r such that

$r > 0$ and f is differentiable on $]x_0 - r, x_0[$ and g is differentiable on $]x_0 - r, x_0[$ and $]x_0 - r, x_0[\subseteq \mathrm{dom}(\frac{f}{g})$ and $[x_0 - r, x_0] \subseteq \mathrm{dom}(\frac{f'_{\restriction]x_0 - r, x_0[}}{g'_{\restriction]x_0 - r, x_0[}})$ and $f(x_0) = 0$ and $g(x_0) = 0$ and f is continuous in x_0 and g is continuous in x_0 and $\frac{f'_{\restriction]x_0 - r, x_0[}}{g'_{\restriction]x_0 - r, x_0[}}$ is left convergent in x_0. Then $\frac{f}{g}$ is left convergent in x_0 and there exists r such that $r > 0$ and $\lim_{x_0^-}(\frac{f}{g}) = \lim_{x_0^-}(\frac{f'_{\restriction]x_0 - r, x_0[}}{g'_{\restriction]x_0 - r, x_0[}})$.

(9) Given a neighbourhood N of x_0 such that

f is differentiable on N and g is differentiable on N and $N \setminus \{x_0\} \subseteq \mathrm{dom}(\frac{f}{g})$ and $N \subseteq \mathrm{dom}(\frac{f'_{\restriction N}}{g'_{\restriction N}})$ and $f(x_0) = 0$ and $g(x_0) = 0$ and $\frac{f'_{\restriction N}}{g'_{\restriction N}}$ is convergent in x_0. Then $\frac{f}{g}$ is convergent in x_0 and there exists a neighbourhood N of x_0 such that $\lim_{x_0}(\frac{f}{g}) = \lim_{x_0}(\frac{f'_{\restriction N}}{g'_{\restriction N}})$.

(10) Given a neighbourhood N of x_0 such that

f is differentiable on N and g is differentiable on N and $N \setminus \{x_0\} \subseteq \mathrm{dom}(\frac{f}{g})$ and $N \subseteq \mathrm{dom}(\frac{f'_{\restriction N}}{g'_{\restriction N}})$ and $f(x_0) = 0$ and $g(x_0) = 0$ and $\frac{f'_{\restriction N}}{g'_{\restriction N}}$ is continuous in x_0. Then $\frac{f}{g}$ is convergent in x_0 and $\lim_{x_0}(\frac{f}{g}) = \frac{f'(x_0)}{g'(x_0)}$.

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