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On the Segmentation of a Simple Closed Curve[1]

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Summary. The main goal of the work was to introduce the concept of the segmentation of a simple closed curve into (arbitrary small) arcs. The existence of it has been proved by Yatsuka Nakamura [21]. The concept of the gap of a segmentation is also introduced. It is the smallest distance between disjoint segments in the segmentation. For this purpose, the relationship between segments of an arc [24] and segments on a simple closed curve [21] has been shown.

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The articles [29], [35], [10], [3], [2], [32], [1], [13], [8], [9], [7], [4], [34], [25], [33], [22], [20], [28], [15], [26], [27], [18], [6], [12], [30], [19], [14], [16], [17], [23], [5], [24], [21], [11], and [31] provide the notation and terminology for this paper.

1. PRELIMINARIES

The scheme *AndScheme* deals with a non empty set $\mathcal{A}$ and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{a; a$ ranges over elements of $\mathcal{A} : \mathcal{P}[a] \ \wedge\ \mathcal{Q}[a]\} = \{a_1; a_1$ ranges over elements of $\mathcal{A} : \mathcal{P}[a_1]\} \cap \{a_2; a_2$ ranges over elements of $\mathcal{A} : \mathcal{Q}[a_2]\}$

for all values of the parameters.

For simplicity, we adopt the following rules: C is a simple closed curve, p, q are points of $\mathcal{E}^2_{\mathrm{T}}$, i, j, k, n are natural numbers, and e is a real number.

Next we state the proposition

(1) For all finite non empty subsets A, B of $\mathbb{R}$ holds $\min(A \cup B) = \min(\min A, \min B)$.

Let T be a non empty topological space. Note that there exists a subset of T which is compact and non empty.

The following propositions are true:

(2) Let T be a non empty topological space, f be a continuous real map of T, and A be a compact subset of T. Then $f^\circ A$ is compact.

(3) For every compact subset A of $\mathbb{R}$ and for every non empty subset B of $\mathbb{R}$ such that $B \subseteq A$ holds $\inf B \in A$.

(4) Let A, B be compact non empty subsets of $\mathcal{E}^n_{\mathrm{T}}$, f be a continuous real map of $[: \mathcal{E}^n_{\mathrm{T}}, \mathcal{E}^n_{\mathrm{T}} :]$, and g be a real map of $\mathcal{E}^n_{\mathrm{T}}$. Suppose that for every point p of $\mathcal{E}^n_{\mathrm{T}}$ there exists a subset G of $\mathbb{R}$ such that $G = \{f(p, q); q$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $q \in B\}$ and $g(p) = \inf G$. Then $\inf(f^\circ[: A, B :]) = \inf(g^\circ A)$.

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(5) Let A, B be compact non empty subsets of $\mathcal{E}^n_{\mathrm{T}}$, f be a continuous real map of $[: \mathcal{E}^n_{\mathrm{T}}, \mathcal{E}^n_{\mathrm{T}} :]$, and g be a real map of $\mathcal{E}^n_{\mathrm{T}}$. Suppose that for every point q of $\mathcal{E}^n_{\mathrm{T}}$ there exists a subset G of $\mathbb{R}$ such that $G = \{f(p, q); p$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $p \in A\}$ and $g(q) = \inf G$. Then $\inf(f^\circ[:A, B:]) = \inf(g^\circ B)$.

(6) If $q \in \mathrm{LowerArc}(C)$ and $q \neq \mathrm{W_{min}}(C)$, then $\mathrm{E_{max}}(C) \leq_C q$.

(7) If $q \in \mathrm{UpperArc}(C)$, then $q \leq_C \mathrm{E_{max}}(C)$.

2. The Euclidean Distance

Let us consider n. The functor $\mathrm{EuclDist}(n)$ yielding a real map of $[: \mathcal{E}^n_{\mathrm{T}}, \mathcal{E}^n_{\mathrm{T}} :]$ is defined as follows:

(Def. 1) For all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ holds $(\mathrm{EuclDist}(n))(p, q) = |p - q|$.

Let T be a non empty topological space and let f be a real map of T. Let us observe that f is continuous if and only if:

(Def. 2) For every point p of T and for every neighbourhood N of $f(p)$ there exists a neighbourhood V of p such that $f^\circ V \subseteq N$.

Let us consider n. Note that $\mathrm{EuclDist}(n)$ is continuous.

3. On the Distance between Subsets of a Euclidean Space

The following proposition is true

(8) For all non empty compact subsets A, B of $\mathcal{E}^n_{\mathrm{T}}$ such that A misses B holds $\mathrm{dist_{min}}(A, B) > 0$.

4. On the Segments

Next we state a number of propositions:

(9) If $p \leq_C q$ and $q \leq_C \mathrm{E_{max}}(C)$ and $p \neq q$, then $\mathrm{Segment}(p, q, C) = \mathrm{Segment}(\mathrm{UpperArc}(C), \mathrm{W_{min}}(C), \mathrm{E_{max}}(C), p, q)$.

(10) If $\mathrm{E_{max}}(C) \leq_C q$, then $\mathrm{Segment}(\mathrm{E_{max}}(C), q, C) = \mathrm{Segment}(\mathrm{LowerArc}(C), \mathrm{E_{max}}(C), \mathrm{W_{min}}(C), \mathrm{E_{max}}(C), q)$.

(11) If $\mathrm{E_{max}}(C) \leq_C q$, then $\mathrm{Segment}(q, \mathrm{W_{min}}(C), C) = \mathrm{Segment}(\mathrm{LowerArc}(C), \mathrm{E_{max}}(C), \mathrm{W_{min}}(C), q, \mathrm{W_{min}}(C))$.

(12) If $p \leq_C q$ and $\mathrm{E_{max}}(C) \leq_C p$, then $\mathrm{Segment}(p, q, C) = \mathrm{Segment}(\mathrm{LowerArc}(C), \mathrm{E_{max}}(C), \mathrm{W_{min}}(C), p, q)$.

(13) If $p \leq_C \mathrm{E_{max}}(C)$ and $\mathrm{E_{max}}(C) \leq_C q$, then $\mathrm{Segment}(p, q, C) = \mathrm{RSegment}(\mathrm{UpperArc}(C), \mathrm{W_{min}}(C), \mathrm{E_{max}}(C), p) \cup \mathrm{LSegment}(\mathrm{LowerArc}(C), \mathrm{E_{max}}(C), \mathrm{W_{min}}(C), q)$.

(14) If $p \leq_C \mathrm{E_{max}}(C)$, then $\mathrm{Segment}(p, \mathrm{W_{min}}(C), C) = \mathrm{RSegment}(\mathrm{UpperArc}(C), \mathrm{W_{min}}(C), \mathrm{E_{max}}(C), p) \cup \mathrm{LSegment}(\mathrm{LowerArc}(C), \mathrm{E_{max}}(C), \mathrm{W_{min}}(C), \mathrm{W_{min}}(C))$.

(15) $\mathrm{RSegment}(\mathrm{UpperArc}(C), \mathrm{W_{min}}(C), \mathrm{E_{max}}(C), p) = \mathrm{Segment}(\mathrm{UpperArc}(C), \mathrm{W_{min}}(C), \mathrm{E_{max}}(C), p, \mathrm{E_{max}}(C))$.

(16) $\mathrm{LSegment}(\mathrm{LowerArc}(C), \mathrm{E_{max}}(C), \mathrm{W_{min}}(C), p) = \mathrm{Segment}(\mathrm{LowerArc}(C), \mathrm{E_{max}}(C), \mathrm{W_{min}}(C), \mathrm{E_{max}}(C), p)$.

(17) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in C$ and $p \neq \mathrm{W_{min}}(C)$ holds $\mathrm{Segment}(p, \mathrm{W_{min}}(C), C)$ is an arc from p to $\mathrm{W_{min}}(C)$.

(18) For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq q$ and $p \leq_C q$ holds $\mathrm{Segment}(p, q, C)$ is an arc from p to q.

(19) $C = \mathrm{Segment}(\mathrm{W_{min}}(C), \mathrm{W_{min}}(C), C)$.

(20) For every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in C$ holds $\mathrm{Segment}(q, \mathrm{W_{min}}(C), C)$ is compact.

(21) For all points q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $q_1 \leq_C q_2$ holds $\mathrm{Segment}(q_1, q_2, C)$ is compact.

5. The Concept of a Segmentation

Let us consider C. A finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is said to be a segmentation of C if it satisfies the conditions (Def. 3).

(Def. 3) $\mathrm{It}_1 = \mathrm{W}_{\min}(C)$ and it is one-to-one and $8 \leq \mathrm{len}\,\mathrm{it}$ and $\mathrm{rng}\,\mathrm{it} \subseteq C$ and for every natural number i such that $1 \leq i$ and $i < \mathrm{len}\,\mathrm{it}$ holds $\mathrm{it}_i \leq_C \mathrm{it}_{i+1}$ and for every natural number i such that $1 \leq i$ and $i+1 < \mathrm{len}\,\mathrm{it}$ holds $\mathrm{Segment}(\mathrm{it}_i, \mathrm{it}_{i+1}, C) \cap \mathrm{Segment}(\mathrm{it}_{i+1}, \mathrm{it}_{i+2}, C) = \{\mathrm{it}_{i+1}\}$ and $\mathrm{Segment}(\mathrm{it}_{\mathrm{len}\,\mathrm{it}}, \mathrm{it}_1, C) \cap \mathrm{Segment}(\mathrm{it}_1, \mathrm{it}_2, C) = \{\mathrm{it}_1\}$ and $\mathrm{Segment}(\mathrm{it}_{\mathrm{len}\,\mathrm{it}-'1}, \mathrm{it}_{\mathrm{len}\,\mathrm{it}}, C) \cap \mathrm{Segment}(\mathrm{it}_{\mathrm{len}\,\mathrm{it}}, \mathrm{it}_1, C) = \{\mathrm{it}_{\mathrm{len}\,\mathrm{it}}\}$ and $\mathrm{Segment}(\mathrm{it}_{\mathrm{len}\,\mathrm{it}-'1}, \mathrm{it}_{\mathrm{len}\,\mathrm{it}}, C)$ misses $\mathrm{Segment}(\mathrm{it}_1, \mathrm{it}_2, C)$ and for all natural numbers i, j such that $1 \leq i$ and $i < j$ and $j < \mathrm{len}\,\mathrm{it}$ and i and j are not adjacent holds $\mathrm{Segment}(\mathrm{it}_i, \mathrm{it}_{i+1}, C)$ misses $\mathrm{Segment}(\mathrm{it}_j, \mathrm{it}_{j+1}, C)$ and for every natural number i such that $1 < i$ and $i+1 < \mathrm{len}\,\mathrm{it}$ holds $\mathrm{Segment}(\mathrm{it}_{\mathrm{len}\,\mathrm{it}}, \mathrm{it}_1, C)$ misses $\mathrm{Segment}(\mathrm{it}_i, \mathrm{it}_{i+1}, C)$.

Let us consider C. One can verify that every segmentation of C is non trivial.

Next we state the proposition

(22) For every segmentation S of C and for every i such that $1 \leq i$ and $i \leq \mathrm{len}\,S$ holds $S_i \in C$.

6. The Segments of a Segmentation

Let us consider C, let i be a natural number, and let S be a segmentation of C. The functor $\mathrm{Segm}(S, i)$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 4) $$\mathrm{Segm}(S, i) = \begin{cases} \mathrm{Segment}(S_i, S_{i+1}, C), & \text{if } 1 \leq i \text{ and } i < \mathrm{len}\,S, \\ \mathrm{Segment}(S_{\mathrm{len}\,S}, S_1, C), & \text{otherwise.} \end{cases}$$

The following proposition is true

(23) For every segmentation S of C such that $i \in \mathrm{dom}\,S$ holds $\mathrm{Segm}(S, i) \subseteq C$.

Let us consider C, let S be a segmentation of C, and let us consider i. Note that $\mathrm{Segm}(S, i)$ is non empty and compact.

The following propositions are true:

(24) For every segmentation S of C and for every p such that $p \in C$ there exists a natural number i such that $i \in \mathrm{dom}\,S$ and $p \in \mathrm{Segm}(S, i)$.

(25) Let S be a segmentation of C and given i, j. Suppose $1 \leq i$ and $i < j$ and $j < \mathrm{len}\,S$ and i and j are not adjacent. Then $\mathrm{Segm}(S, i)$ misses $\mathrm{Segm}(S, j)$.

(26) For every segmentation S of C and for every j such that $1 < j$ and $j < \mathrm{len}\,S -' 1$ holds $\mathrm{Segm}(S, \mathrm{len}\,S)$ misses $\mathrm{Segm}(S, j)$.

(27) Let S be a segmentation of C and given i, j. Suppose $1 \leq i$ and $i < j$ and $j < \mathrm{len}\,S$ and i and j are adjacent. Then $\mathrm{Segm}(S, i) \cap \mathrm{Segm}(S, j) = \{S_{i+1}\}$.

(28) Let S be a segmentation of C and given i, j. Suppose $1 \leq i$ and $i < j$ and $j < \mathrm{len}\,S$ and i and j are adjacent. Then $\mathrm{Segm}(S, i)$ meets $\mathrm{Segm}(S, j)$.

(29) For every segmentation S of C holds $\mathrm{Segm}(S, \mathrm{len}\,S) \cap \mathrm{Segm}(S, 1) = \{S_1\}$.

(30) For every segmentation S of C holds $\mathrm{Segm}(S, \mathrm{len}\,S)$ meets $\mathrm{Segm}(S, 1)$.

(31) For every segmentation S of C holds $\mathrm{Segm}(S, \mathrm{len}\,S) \cap \mathrm{Segm}(S, \mathrm{len}\,S -' 1) = \{S_{\mathrm{len}\,S}\}$.

(32) For every segmentation S of C holds $\mathrm{Segm}(S, \mathrm{len}\,S)$ meets $\mathrm{Segm}(S, \mathrm{len}\,S -' 1)$.

7. The Diameter of a Segmentation

Let us consider n and let C be a subset of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\varnothing C$ yields a real number and is defined by:

(Def. 5) There exists a subset W of $\mathcal{E}^n$ such that $W = C$ and $\varnothing C = \varnothing W$.

Let us consider C and let S be a segmentation of C. The functor $\varnothing S$ yields a real number and is defined by:

(Def. 6) There exists a non empty finite subset S_1 of $\mathbb{R}$ such that $S_1 = \{\varnothing \mathrm{Segm}(S,i) : i \in \mathrm{dom} S\}$ and $\varnothing S = \max S_1$.

One can prove the following three propositions:

(33) For every segmentation S of C and for every i holds $\varnothing \mathrm{Segm}(S,i) \leq \varnothing S$.

(34) For every segmentation S of C and for every real number e such that for every i holds $\varnothing \mathrm{Segm}(S,i) < e$ holds $\varnothing S < e$.

(35) For every real number e such that $e > 0$ there exists a segmentation S of C such that $\varnothing S < e$.

8. The Concept of the Gap of a Segmentation

Let us consider C and let S be a segmentation of C. The functor $\mathrm{Gap}(S)$ yields a real number and is defined by the condition (Def. 7).

(Def. 7) There exist non empty finite subsets S_1, S_2 of $\mathbb{R}$ such that $S_1 = \{\mathrm{dist}_{\min}(\mathrm{Segm}(S,i), \mathrm{Segm}(S,j)) : 1 \leq i \ \wedge \ i < j \ \wedge \ j < \mathrm{len} S \ \wedge \ i$ and j are not adjacent$\}$ and $S_2 = \{\mathrm{dist}_{\min}(\mathrm{Segm}(S, \mathrm{len} S), \mathrm{Segm}(S,k)) : 1 < k \wedge k < \mathrm{len} S -' 1\}$ and $\mathrm{Gap}(S) = \min(\min S_1, \min S_2)$.

The following two propositions are true:

(36) Let S be a segmentation of C. Then there exists a finite non empty subset F of $\mathbb{R}$ such that $F = \{\mathrm{dist}_{\min}(\mathrm{Segm}(S,i), \mathrm{Segm}(S,j)) : 1 \leq i \ \wedge \ i < j \ \wedge \ j \leq \mathrm{len} S \ \wedge \ \mathrm{Segm}(S,i)$ misses $\mathrm{Segm}(S,j)\}$ and $\mathrm{Gap}(S) = \min F$.

(37) For every segmentation S of C holds $\mathrm{Gap}(S) > 0$.

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