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Cages - the External Approximation of Jordan's Curve

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Summary. On the Euclidean plane Jordan's curve may be approximated with a polygonal path of sides parallel to coordinate axes, either externally, or internally. The paper deals with the external approximation, and the existence of a *Cage* – an external polygonal path – is proved.

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The articles [25], [30], [21], [3], [27], [2], [22], [31], [6], [7], [1], [4], [8], [29], [9], [18], [16], [17], [24], [32], [13], [23], [5], [14], [15], [19], [20], [12], [28], [26], [10], and [11] provide the notation and terminology for this paper.

1. Generalities

We adopt the following convention: i, j, k, n are natural numbers, D is a non empty set, and f, g are finite sequences of elements of D.

One can prove the following four propositions:

(3)[1] Let T be a non empty topological space and B, C_1, C_2, D be subsets of T. Suppose B is connected and C_1 is a component of D and C_2 is a component of D and B meets C_1 and B meets C_2 and $B \subseteq D$. Then $C_1 = C_2$.

(4) If for every n holds $f \restriction n = g \restriction n$, then $f = g$.

(5) If $n \in \operatorname{dom} f$, then there exists k such that $k \in \operatorname{dom} \operatorname{Rev}(f)$ and $n + k = \operatorname{len} f + 1$ and $f_n = (\operatorname{Rev}(f))_k$.

(6) If $n \in \operatorname{dom} \operatorname{Rev}(f)$, then there exists k such that $k \in \operatorname{dom} f$ and $n + k = \operatorname{len} f + 1$ and $(\operatorname{Rev}(f))_n = f_k$.

2. Go-Board Preliminaries

For simplicity, we adopt the following convention: G is a Go-board, f, g are finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, p is a point of $\mathcal{E}^2_{\mathrm{T}}$, r, s are real numbers, and x is a set.

We now state a number of propositions:

(7) Let D be a non empty set, G be a matrix over D, and f be a finite sequence of elements of D. Then f is a sequence which elements belong to G if and only if $\operatorname{Rev}(f)$ is a sequence which elements belong to G.

[1] The propositions (1) and (2) have been removed.

(8) Let G be a matrix over $\mathcal{E}^2_{\mathrm{T}}$ and f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a sequence which elements belong to G and $1 \leq k$ and $k \leq \operatorname{len} f$. Then $f_k \in \operatorname{Values} G$.

(9) If $n \leq \operatorname{len} f$ and $x \in \widetilde{\mathcal{L}}(f_{\lfloor n})$, then there exists a natural number i such that $n+1 \leq i$ and $i+1 \leq \operatorname{len} f$ and $x \in \mathcal{L}(f,i)$.

(10) If f is a sequence which elements belong to G and $1 \leq k$ and $k+1 \leq \operatorname{len} f$, then $f_k \in \operatorname{left\_cell}(f,k,G)$ and $f_k \in \operatorname{right\_cell}(f,k,G)$.

(11) If f is a sequence which elements belong to G and $1 \leq k$ and $k+1 \leq \operatorname{len} f$, then $\operatorname{Int} \operatorname{left\_cell}(f,k,G) \neq \emptyset$ and $\operatorname{Int} \operatorname{right\_cell}(f,k,G) \neq \emptyset$.

(12) Suppose f is a sequence which elements belong to G and $1 \leq k$ and $k+1 \leq \operatorname{len} f$. Then $\operatorname{Int} \operatorname{left\_cell}(f,k,G)$ is connected and $\operatorname{Int} \operatorname{right\_cell}(f,k,G)$ is connected.

(13) If f is a sequence which elements belong to G and $1 \leq k$ and $k+1 \leq \operatorname{len} f$, then $\overline{\operatorname{Int} \operatorname{left\_cell}(f,k,G)} = \operatorname{left\_cell}(f,k,G)$ and $\overline{\operatorname{Int} \operatorname{right\_cell}(f,k,G)} = \operatorname{right\_cell}(f,k,G)$.

(14) Suppose f is a sequence which elements belong to G and $\mathcal{L}(f,k)$ is horizontal. Then there exists j such that $1 \leq j$ and $j \leq \operatorname{width} G$ and for every p such that $p \in \mathcal{L}(f,k)$ holds $p_{\mathbf{2}} = (G \circ (1,j))_{\mathbf{2}}$.

(15) Suppose f is a sequence which elements belong to G and $\mathcal{L}(f,k)$ is vertical. Then there exists i such that $1 \leq i$ and $i \leq \operatorname{len} G$ and for every p such that $p \in \mathcal{L}(f,k)$ holds $p_{\mathbf{1}} = (G \circ (i,1))_{\mathbf{1}}$.

(16) If f is a sequence which elements belong to G and special and $i \leq \operatorname{len} G$ and $j \leq \operatorname{width} G$, then $\operatorname{Int} \operatorname{cell}(G,i,j)$ misses $\widetilde{\mathcal{L}}(f)$.

(17) Suppose f is a sequence which elements belong to G and special and $1 \leq k$ and $k+1 \leq \operatorname{len} f$. Then $\operatorname{Int} \operatorname{left\_cell}(f,k,G)$ misses $\widetilde{\mathcal{L}}(f)$ and $\operatorname{Int} \operatorname{right\_cell}(f,k,G)$ misses $\widetilde{\mathcal{L}}(f)$.

(18) Suppose $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$. Then $(G \circ (i,j))_{\mathbf{1}} = (G \circ (i,j+1))_{\mathbf{1}}$ and $(G \circ (i,j))_{\mathbf{2}} = (G \circ (i+1,j))_{\mathbf{2}}$ and $(G \circ (i+1,j+1))_{\mathbf{1}} = (G \circ (i+1,j))_{\mathbf{1}}$ and $(G \circ (i+1,j+1))_{\mathbf{2}} = (G \circ (i,j+1))_{\mathbf{2}}$.

(19) Let i, j be natural numbers. Suppose $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$. Then $p \in \operatorname{cell}(G,i,j)$ if and only if the following conditions are satisfied:

(i) $(G \circ (i,j))_{\mathbf{1}} \leq p_{\mathbf{1}}$,

(ii) $p_{\mathbf{1}} \leq (G \circ (i+1,j))_{\mathbf{1}}$,

(iii) $(G \circ (i,j))_{\mathbf{2}} \leq p_{\mathbf{2}}$, and

(iv) $p_{\mathbf{2}} \leq (G \circ (i,j+1))_{\mathbf{2}}$.

(20) If $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$, then $\operatorname{cell}(G,i,j) = \{[r,s] : (G \circ (i,j))_{\mathbf{1}} \leq r \wedge r \leq (G \circ (i+1,j))_{\mathbf{1}} \wedge (G \circ (i,j))_{\mathbf{2}} \leq s \wedge s \leq (G \circ (i,j+1))_{\mathbf{2}}\}$.

(21) Suppose $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$ and $p \in \operatorname{Values} G$ and $p \in \operatorname{cell}(G,i,j)$. Then $p = G \circ (i,j)$ or $p = G \circ (i,j+1)$ or $p = G \circ (i+1,j+1)$ or $p = G \circ (i+1,j)$.

(22) If $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$, then $G \circ (i,j) \in \operatorname{cell}(G,i,j)$ and $G \circ (i,j+1) \in \operatorname{cell}(G,i,j)$ and $G \circ (i+1,j+1) \in \operatorname{cell}(G,i,j)$ and $G \circ (i+1,j) \in \operatorname{cell}(G,i,j)$.

(23) If $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$ and $p \in \operatorname{Values} G$ and $p \in \operatorname{cell}(G,i,j)$, then p is extremal in $\operatorname{cell}(G,i,j)$.

(24) Suppose $2 \leq \operatorname{len} G$ and $2 \leq \operatorname{width} G$ and f is a sequence which elements belong to G and $1 \leq k$ and $k+1 \leq \operatorname{len} f$. Then there exist i, j such that $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$ and $\mathcal{L}(f,k) \subseteq \operatorname{cell}(G,i,j)$.

(25) Suppose $2 \leq \operatorname{len} G$ and $2 \leq \operatorname{width} G$ and f is a sequence which elements belong to G and $1 \leq k$ and $k+1 \leq \operatorname{len} f$ and $p \in \operatorname{Values} G$ and $p \in \mathcal{L}(f,k)$. Then $p = f_k$ or $p = f_{k+1}$.

(26) If $\langle i, j\rangle \in$ the indices of G and $1 \leq k$ and $k \leq \operatorname{width} G$, then $(G \circ (i, j))_{\mathbf{1}} \leq (G \circ (\operatorname{len} G, k))_{\mathbf{1}}$.

(27) If $\langle i, j\rangle \in$ the indices of G and $1 \leq k$ and $k \leq \operatorname{len} G$, then $(G \circ (i, j))_{\mathbf{2}} \leq (G \circ (k, \operatorname{width} G))_{\mathbf{2}}$.

(28) Suppose f is a sequence which elements belong to G and special and $\widetilde{\mathcal{L}}(g) \subseteq \widetilde{\mathcal{L}}(f)$ and $1 \leq k$ and $k+1 \leq \operatorname{len} f$. Let A be a subset of $\mathcal{E}^2_{\mathrm{T}}$. If $A = \mathrm{right\_cell}(f, k, G) \setminus \widetilde{\mathcal{L}}(g)$ or $A = \mathrm{left\_cell}(f, k, G) \setminus \widetilde{\mathcal{L}}(g)$, then A is connected.

(29) Let f be a non constant standard special circular sequence. Suppose f is a sequence which elements belong to G. Let given k. If $1 \leq k$ and $k+1 \leq \operatorname{len} f$, then $\mathrm{right\_cell}(f, k, G) \setminus \widetilde{\mathcal{L}}(f) \subseteq \mathrm{RightComp}(f)$ and $\mathrm{left\_cell}(f, k, G) \setminus \widetilde{\mathcal{L}}(f) \subseteq \mathrm{LeftComp}(f)$.

3. CAGES

In the sequel C denotes a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and i_1, i_2 denote natural numbers.

One can prove the following propositions:

(30) There exists i such that $1 \leq i$ and $i+1 \leq \operatorname{len} \mathrm{Gauge}(C, n)$ and $\mathrm{N}_{\min}(C) \in \mathrm{cell}(\mathrm{Gauge}(C, n), i, \operatorname{width} \mathrm{Gauge}(C, n) -' 1)$ and $\mathrm{N}_{\min}(C) \neq \mathrm{Gauge}(C, n) \circ (i, \operatorname{width} \mathrm{Gauge}(C, n) -' 1)$.

(31) Suppose that $1 \leq i_1$ and $i_1 + 1 \leq \operatorname{len} \mathrm{Gauge}(C, n)$ and $\mathrm{N}_{\min}(C) \in \mathrm{cell}(\mathrm{Gauge}(C, n), i_1, \operatorname{width} \mathrm{Gauge}(C, n) -' 1)$ and $\mathrm{N}_{\min}(C) \neq \mathrm{Gauge}(C, n) \circ (i_1, \operatorname{width} \mathrm{Gauge}(C, n) -' 1)$ and $1 \leq i_2$ and $i_2 + 1 \leq \operatorname{len} \mathrm{Gauge}(C, n)$ and $\mathrm{N}_{\min}(C) \in \mathrm{cell}(\mathrm{Gauge}(C, n), i_2, \operatorname{width} \mathrm{Gauge}(C, n) -' 1)$ and $\mathrm{N}_{\min}(C) \neq \mathrm{Gauge}(C, n) \circ (i_2, \operatorname{width} \mathrm{Gauge}(C, n) -' 1)$. Then $i_1 = i_2$.

(32) Let f be a standard non constant special circular sequence. Suppose that

(i) f is a sequence which elements belong to $\mathrm{Gauge}(C, n)$,

(ii) for every k such that $1 \leq k$ and $k+1 \leq \operatorname{len} f$ holds $\mathrm{left\_cell}(f, k, \mathrm{Gauge}(C, n))$ misses C and $\mathrm{right\_cell}(f, k, \mathrm{Gauge}(C, n))$ meets C, and

(iii) there exists i such that $1 \leq i$ and $i+1 \leq \operatorname{len} \mathrm{Gauge}(C, n)$ and $f_1 = \mathrm{Gauge}(C, n) \circ (i, \operatorname{width} \mathrm{Gauge}(C, n))$ and $f_2 = \mathrm{Gauge}(C, n) \circ (i+1, \operatorname{width} \mathrm{Gauge}(C, n))$ and $\mathrm{N}_{\min}(C) \in \mathrm{cell}(\mathrm{Gauge}(C, n), i, \operatorname{width} \mathrm{Gauge}(C, n) -' 1)$ and $\mathrm{N}_{\min}(C) \neq \mathrm{Gauge}(C, n) \circ (i, \operatorname{width} \mathrm{Gauge}(C, n) -' 1)$.

Then $\mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f)) = f_1$.

Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. Let us assume that C is connected. The functor $\mathrm{Cage}(C, n)$ yields a clockwise oriented standard non constant special circular sequence and is defined by the conditions (Def. 1).

(Def. 1)(i) $\mathrm{Cage}(C, n)$ is a sequence which elements belong to $\mathrm{Gauge}(C, n)$,

(ii) there exists i such that $1 \leq i$ and $i+1 \leq \operatorname{len} \mathrm{Gauge}(C, n)$ and $(\mathrm{Cage}(C, n))_1 = \mathrm{Gauge}(C, n) \circ (i, \operatorname{width} \mathrm{Gauge}(C, n))$ and $(\mathrm{Cage}(C, n))_2 = \mathrm{Gauge}(C, n) \circ (i+1, \operatorname{width} \mathrm{Gauge}(C, n))$ and $\mathrm{N}_{\min}(C) \in \mathrm{cell}(\mathrm{Gauge}(C, n), i, \operatorname{width} \mathrm{Gauge}(C, n) -' 1)$ and $\mathrm{N}_{\min}(C) \neq \mathrm{Gauge}(C, n) \circ (i, \operatorname{width} \mathrm{Gauge}(C, n) -' 1)$, and

(iii) for every k such that $1 \leq k$ and $k+2 \leq \operatorname{len} \mathrm{Cage}(C, n)$ holds if $\mathrm{front\_left\_cell}(\mathrm{Cage}(C, n), k, \mathrm{Gauge}(C, n))$ misses C and $\mathrm{front\_right\_cell}(\mathrm{Cage}(C, n), k, \mathrm{Gauge}(C, n))$ misses C, then $\mathrm{Cage}(C, n)$ turns right k, $\mathrm{Gauge}(C, n)$ and if $\mathrm{front\_left\_cell}(\mathrm{Cage}(C, n), k, \mathrm{Gauge}(C, n))$ misses C and $\mathrm{front\_right\_cell}(\mathrm{Cage}(C, n), k, \mathrm{Gauge}(C, n))$ meets C, then $\mathrm{Cage}(C, n)$ goes straight k, $\mathrm{Gauge}(C, n)$ and if $\mathrm{front\_left\_cell}(\mathrm{Cage}(C, n), k, \mathrm{Gauge}(C, n))$ meets C, then $\mathrm{Cage}(C, n)$ turns left k, $\mathrm{Gauge}(C, n)$.

Next we state two propositions:

(33) If C is connected and $1 \leq k$ and $k+1 \leq \operatorname{len} \mathrm{Cage}(C, n)$, then $\mathrm{left\_cell}(\mathrm{Cage}(C, n), k, \mathrm{Gauge}(C, n))$ misses C and $\mathrm{right\_cell}(\mathrm{Cage}(C, n), k, \mathrm{Gauge}(C, n))$ meets C.

(34) If C is connected, then $\mathrm{N}_{\min}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))) = (\mathrm{Cage}(C, n))_1$.

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