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The Ordering of Points on a Curve. Part II

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Summary. The proof of the Jordan Curve Theorem according to [11] is continued. The notions of the first and last point of a oriented arc are introduced as well as ordering of points on a curve in $\mathcal{E}^2_T$.

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The articles [12], [14], [1], [15], [2], [3], [4], [7], [13], [9], [8], [10], [5], and [6] provide the notation and terminology for this paper.

1. First and last point of a curve

One can prove the following proposition

(1) Let P, Q be subsets of $\mathcal{E}^2_{\rm T}$, p_1, p_2, q_1 be points of $\mathcal{E}^2_{\rm T}$, f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P$, and s_1 be a real number. Suppose that P is an arc from p_1 to p_2 and $q_1 \in P$ and $q_1 \in Q$ and $f(s_1) = q_1$ and f is a homeomorphism and $f(0) = p_1$ and $f(1) = p_2$ and $0 \leq s_1$ and $s_1 \leq 1$ and for every real number t such that $0 \leq t$ and $t < s_1$ holds $f(t) \notin Q$. Let g be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P$ and s_2 be a real number. Suppose g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) = q_1$ and $0 \leq s_2$ and $s_2 \leq 1$. Let t be a real number. If $0 \leq t$ and $t < s_2$, then $g(t) \notin Q$.

Let P, Q be subsets of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Let us assume that P meets Q and $P \cap Q$ is closed and P is an arc from p_1 to p_2. The functor $\mathrm{FPoint}(P, p_1, p_2, Q)$ yields a point of $\mathcal{E}^2_{\rm T}$ and is defined by the conditions (Def. 1).

(Def. 1)(i) $\mathrm{FPoint}(P, p_1, p_2, Q) \in P \cap Q$, and

(ii) for every map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P$ and for every real number s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) = \mathrm{FPoint}(P, p_1, p_2, Q)$ and $0 \leq s_2$ and $s_2 \leq 1$ and for every real number t such that $0 \leq t$ and $t < s_2$ holds $g(t) \notin Q$.

The following propositions are true:

(2) Let P, Q be subsets of $\mathcal{E}^2_{\rm T}$ and p, p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If $p \in P$ and P is an arc from p_1 to p_2 and $Q = \{p\}$, then $\mathrm{FPoint}(P, p_1, p_2, Q) = p$.

(3) Let P be a subset of $\mathcal{E}^2_{\rm T}$, Q be a subset of $\mathcal{E}^2_{\rm T}$, and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If $p_1 \in Q$ and $P \cap Q$ is closed and P is an arc from p_1 to p_2, then $\mathrm{FPoint}(P, p_1, p_2, Q) = p_1$.

(4) Let P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$, p_1, p_2, q_1 be points of $\mathcal{E}^2_{\mathrm{T}}$, f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction P$, and s_1 be a real number. Suppose that P is an arc from p_1 to p_2 and $q_1 \in P$ and $q_1 \in Q$ and $f(s_1) = q_1$ and f is a homeomorphism and $f(0) = p_1$ and $f(1) = p_2$ and $0 \leq s_1$ and $s_1 \leq 1$ and for every real number t such that $1 \geq t$ and $t > s_1$ holds $f(t) \notin Q$. Let g be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction P$ and s_2 be a real number. Suppose g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) = q_1$ and $0 \leq s_2$ and $s_2 \leq 1$. Let t be a real number. If $1 \geq t$ and $t > s_2$, then $g(t) \notin Q$.

Let P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Let us assume that P meets Q and $P \cap Q$ is closed and P is an arc from p_1 to p_2. The functor $\mathrm{LPoint}(P, p_1, p_2, Q)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by the conditions (Def. 2).

(Def. 2)(i) $\mathrm{LPoint}(P, p_1, p_2, Q) \in P \cap Q$, and

(ii) for every map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction P$ and for every real number s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) = \mathrm{LPoint}(P, p_1, p_2, Q)$ and $0 \leq s_2$ and $s_2 \leq 1$ and for every real number t such that $1 \geq t$ and $t > s_2$ holds $g(t) \notin Q$.

One can prove the following three propositions:

(5) Let P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and p, p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in P$ and P is an arc from p_1 to p_2 and $Q = \{p\}$, then $\mathrm{LPoint}(P, p_1, p_2, Q) = p$.

(6) Let P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_2 \in Q$ and $P \cap Q$ is closed and P is an arc from p_1 to p_2, then $\mathrm{LPoint}(P, p_1, p_2, Q) = p_2$.

(7) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$, Q be a subset of $\mathcal{E}^2_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P \subseteq Q$ and P is closed and an arc from p_1 to p_2. Then $\mathrm{FPoint}(P, p_1, p_2, Q) = p_1$ and $\mathrm{LPoint}(P, p_1, p_2, Q) = p_2$.

2. The Ordering of Points on a Curve

Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. We say that LE q_1, q_2, P, p_1, p_2 if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $q_1 \in P$,

(ii) $q_2 \in P$, and

(iii) for every map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction P$ and for all real numbers s_1, s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_1) = q_1$ and $0 \leq s_1$ and $s_1 \leq 1$ and $g(s_2) = q_2$ and $0 \leq s_2$ and $s_2 \leq 1$ holds $s_1 \leq s_2$.

Next we state several propositions:

(8) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$, p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$, g be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction P$, and s_1, s_2 be real numbers. Suppose that P is an arc from p_1 to p_2 and g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_1) = q_1$ and $0 \leq s_1$ and $s_1 \leq 1$ and $g(s_2) = q_2$ and $0 \leq s_2$ and $s_2 \leq 1$ and $s_1 \leq s_2$. Then LE q_1, q_2, P, p_1, p_2.

(9) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q_1 \in P$, then LE q_1, q_1, P, p_1, p_2.

(10) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and $q_1 \in P$. Then LE p_1, q_1, P, p_1, p_2 and LE q_1, p_2, P, p_1, p_2.

(11) For every subset P of $\mathcal{E}^2_{\mathrm{T}}$ and for all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that P is an arc from p_1 to p_2 holds LE p_1, p_2, P, p_1, p_2.

(12) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and LE q_1, q_2, P, p_1, p_2 and LE q_2, q_1, P, p_1, p_2. Then $q_1 = q_2$.

(13) Let P be a subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1, q_2, q_3 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and LE q_1, q_2, P, p_1, p_2 and LE q_2, q_3, P, p_1, p_2. Then LE q_1, q_3, P, p_1, p_2.

(14) Let P be a subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and $q_1 \in P$ and $q_2 \in P$ and $q_1 \neq q_2$. Then LE q_1, q_2, P, p_1, p_2 and not LE q_2, q_1, P, p_1, p_2 or LE q_2, q_1, P, p_1, p_2 and not LE q_1, q_2, P, p_1, p_2.

3. Some properties of the ordering of points on a curve

We now state a number of propositions:

(15) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, Q be a subset of $\mathcal{E}^2_{\rm T}$, and q be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $\widetilde{\mathcal{L}}(f) \cap Q$ is closed and $q \in \widetilde{\mathcal{L}}(f)$ and $q \in Q$. Then LE $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q)$, q, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$.

(16) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, Q be a subset of $\mathcal{E}^2_{\rm T}$, and q be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is a special sequence and $\widetilde{\mathcal{L}}(f) \cap Q$ is closed and $q \in \widetilde{\mathcal{L}}(f)$ and $q \in Q$. Then LE q, $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q)$, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$.

(17) For all points q_1, q_2, p_1, p_2 of $\mathcal{E}^2_{\rm T}$ such that $p_1 \neq p_2$ holds if LE q_1, q_2, $\mathcal{L}(p_1, p_2)$, p_1, p_2, then $q_1 \leq_{p_1, p_2} q_2$.

(18) Let P, Q be subsets of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and P meets Q and $P \cap Q$ is closed. Then $\mathrm{FPoint}(P, p_1, p_2, Q) = \mathrm{LPoint}(P, p_2, p_1, Q)$ and $\mathrm{LPoint}(P, p_1, p_2, Q) = \mathrm{FPoint}(P, p_2, p_1, Q)$.

(19) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, Q be a subset of $\mathcal{E}^2_{\rm T}$, and i be a natural number. Suppose $\widetilde{\mathcal{L}}(f)$ meets Q and Q is closed and f is a special sequence and $1 \leq i$ and $i + 1 \leq \mathrm{len}\, f$ and $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) \in \mathcal{L}(f, i)$. Then $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) = \mathrm{FPoint}(\mathcal{L}(f, i), f_i, f_{i+1}, Q)$.

(20) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, Q be a subset of $\mathcal{E}^2_{\rm T}$, and i be a natural number. Suppose $\widetilde{\mathcal{L}}(f)$ meets Q and Q is closed and f is a special sequence and $1 \leq i$ and $i + 1 \leq \mathrm{len}\, f$ and $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) \in \mathcal{L}(f, i)$. Then $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) = \mathrm{LPoint}(\mathcal{L}(f, i), f_i, f_{i+1}, Q)$.

(21) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and i be a natural number. Suppose $1 \leq i$ and $i + 1 \leq \mathrm{len}\, f$ and f is a special sequence and $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, \mathcal{L}(f, i)) \in \mathcal{L}(f, i)$. Then $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, \mathcal{L}(f, i)) = f_i$.

(22) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and i be a natural number. Suppose $1 \leq i$ and $i + 1 \leq \mathrm{len}\, f$ and f is a special sequence and $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, \mathcal{L}(f, i)) \in \mathcal{L}(f, i)$. Then $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, \mathcal{L}(f, i)) = f_{i+1}$.

(23) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and i be a natural number. Suppose f is a special sequence and $1 \leq i$ and $i + 1 \leq \mathrm{len}\, f$. Then LE f_i, f_{i+1}, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$.

(24) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and i, j be natural numbers. Suppose f is a special sequence and $1 \leq i$ and $i \leq j$ and $j \leq \mathrm{len}\, f$. Then LE f_i, f_j, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$.

(25) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, q be a point of $\mathcal{E}^2_{\rm T}$, and i be a natural number. Suppose f is a special sequence and $1 \leq i$ and $i + 1 \leq \mathrm{len}\, f$ and $q \in \mathcal{L}(f, i)$. Then LE f_i, q, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$.

(26) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, q be a point of $\mathcal{E}^2_{\rm T}$, and i be a natural number. Suppose f is a special sequence and $1 \leq i$ and $i + 1 \leq \mathrm{len}\, f$ and $q \in \mathcal{L}(f, i)$. Then LE q, f_{i+1}, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$.

(27) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, Q be a subset of $\mathcal{E}^2_{\mathrm{T}}$, q be a point of $\mathcal{E}^2_{\mathrm{T}}$, and i, j be natural numbers. Suppose that $\widetilde{\mathcal{L}}(f)$ meets Q and f is a special sequence and Q is closed and $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) \in \mathcal{L}(f,i)$ and $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$ and $q \in \mathcal{L}(f,j)$ and $1 \leq j$ and $j+1 \leq \mathrm{len}\, f$ and $q \in Q$ and $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) \neq q$. Then $i \leq j$ and if $i = j$, then $\mathrm{FPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) \leq_{f_i, f_{i+1}} q$.

(28) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, Q be a subset of $\mathcal{E}^2_{\mathrm{T}}$, q be a point of $\mathcal{E}^2_{\mathrm{T}}$, and i, j be natural numbers. Suppose that $\widetilde{\mathcal{L}}(f)$ meets Q and f is a special sequence and Q is closed and $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) \in \mathcal{L}(f,i)$ and $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$ and $q \in \mathcal{L}(f,j)$ and $1 \leq j$ and $j+1 \leq \mathrm{len}\, f$ and $q \in Q$ and $\mathrm{LPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q) \neq q$. Then $i \geq j$ and if $i = j$, then $q \leq_{f_i, f_{i+1}} \mathrm{LPoint}(\widetilde{\mathcal{L}}(f), f_1, f_{\mathrm{len}\, f}, Q)$.

(29) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$, and i be a natural number. Suppose $q_1 \in \mathcal{L}(f,i)$ and $q_2 \in \mathcal{L}(f,i)$ and f is a special sequence and $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$. If LE q_1, q_2, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$, then LE q_1, q_2, $\mathcal{L}(f,i)$, f_i, f_{i+1}.

(30) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $q_1 \in \widetilde{\mathcal{L}}(f)$ and $q_2 \in \widetilde{\mathcal{L}}(f)$ and f is a special sequence and $q_1 \neq q_2$. Then LE q_1, q_2, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\mathrm{len}\, f}$ if and only if for all natural numbers i, j such that $q_1 \in \mathcal{L}(f,i)$ and $q_2 \in \mathcal{L}(f,j)$ and $1 \leq i$ and $i+1 \leq \mathrm{len}\, f$ and $1 \leq j$ and $j+1 \leq \mathrm{len}\, f$ holds $i \leq j$ and if $i = j$, then $q_1 \leq_{f_i, f_{i+1}} q_2$.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[5] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/tops_2.html.

[6] Agata Darmochwał. The Euclidean space. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/euclid.html.

[7] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces — fundamental concepts. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topmetr.html.

[8] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topreal1.html.

[9] Yatsuka Nakamura and Roman Matuszewski. Reconstructions of special sequences. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/jordan3.html.

[10] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[11] Yukio Takeuchi and Yatsuka Nakamura. On the Jordan curve theorem. Technical Report 19804, Dept. of Information Eng., Shinshu University, 500 Wakasato, Nagano city, Japan, April 1980.

[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[13] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[14] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[15] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

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