JOURNAL OF FORMALIZED MATHEMATICS
Volume 9, Released 1997, Published 2003
Inst. of Computer Science, Univ. of Białystok

Subsequences of Standard Special Circular Sequences in $\mathcal{E}^2_{\mathrm{T}}$

Yatsuka Nakamura
Shinshu University
Nagano

Roman Matuszewski
Warsaw University
Białystok

Adam Grabowski
Warsaw University
Białystok

Summary. It is known that a standard special circular sequence in $\mathcal{E}^2_{\mathrm{T}}$ properly defines a special polygon. We are interested in a part of such a sequence. It is shown that if the first point and the last point of the subsequence are different, it becomes a special polygonal sequence. The concept of "a part of" is introduced, and the subsequence having this property can be characterized by using "mid" function. For such subsequences, the concepts of "Upper" and "Lower" parts are introduced.

MML Identifier: JORDAN4.

WWW: `http://mizar.org/JFM/Vol9/jordan4.html`

The articles [15], [8], [1], [13], [18], [2], [3], [17], [4], [6], [7], [10], [12], [14], [5], [16], [9], and [11] provide the notation and terminology for this paper.

1. Preliminaries

In this paper i, i_1, i_2, i_3, j, k, n are natural numbers.

The following propositions are true:

(1) If $n -' i = 0$, then $n \leq i$.

(2) If $i \leq j$, then $(j+k) -' i = (j+k) - i$.

(3) If $i \leq j$, then $(j+k) -' i = (j -' i) + k$.

(4) If $i_1 \neq 0$ and $i_2 = i_3 \cdot i_1$, then $i_3 \leq i_2$.

(5) If $i_1 < i_2$, then $i_1 \div i_2 = 0$.

(6) If $0 < j$ and $j < i$ and $i < j + j$, then $i \bmod j \neq 0$.

(7) If $0 < j$ and $j \leq i$ and $i < j + j$, then $i \bmod j = i - j$ and $i \bmod j = i -' j$.

(8) $(j + j) \bmod j = 0$.

(9) If $0 < k$ and $k \leq j$ and $k \bmod j = 0$, then $k = j$.

2. SOME FACTS ABOUT CUTTING OF FINITE SEQUENCES

In the sequel D denotes a non empty set and f_1 denotes a finite sequence of elements of D.

Next we state a number of propositions:

(14)[1] For every f_1 such that f_1 is circular and $1 \leq \operatorname{len} f_1$ holds $f_1(1) = f_1(\operatorname{len} f_1)$.

(15) For all f_1, i_1, i_2 such that $i_1 \leq i_2$ holds $f_1 \upharpoonright i_1 \upharpoonright i_2 = f_1 \upharpoonright i_1$ and $f_1 \upharpoonright i_2 \upharpoonright i_1 = f_1 \upharpoonright i_1$.

(16) $\varepsilon_D \upharpoonright i = \varepsilon_D$.

(17) $\operatorname{Rev}(\varepsilon_D) = \varepsilon_D$.

(18) For all f_1, k such that $k < \operatorname{len} f_1$ holds $(f_1)_{\downharpoonleft k}(\operatorname{len}((f_1)_{\downharpoonleft k})) = f_1(\operatorname{len} f_1)$ and $((f_1)_{\downharpoonleft k})_{\operatorname{len}((f_1)_{\downharpoonleft k})} = (f_1)_{\operatorname{len} f_1}$.

(19) Let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i. If g is a special sequence and $i+1 < \operatorname{len} g$, then $g_{\downharpoonleft i}$ is a special sequence.

(20) For all f_1, i_1, i_2 such that $1 \leq i_2$ and $i_2 \leq i_1$ and $i_1 \leq \operatorname{len} f_1$ holds $\operatorname{len} \operatorname{mid}(f_1, i_2, i_1) = (i_1 -' i_2) + 1$.

(21) For all f_1, i_1, i_2 such that $1 \leq i_2$ and $i_2 \leq i_1$ and $i_1 \leq \operatorname{len} f_1$ holds $\operatorname{len} \operatorname{mid}(f_1, i_1, i_2) = (i_1 -' i_2) + 1$.

(22) For all f_1, i_1, i_2, j such that $1 \leq i_1$ and $i_1 \leq i_2$ and $i_2 \leq \operatorname{len} f_1$ holds $(\operatorname{mid}(f_1, i_1, i_2))(\operatorname{len} \operatorname{mid}(f_1, i_1, i_2)) = f_1(i_2)$.

(23) For all f_1, i_1, i_2, j such that $1 \leq i_1$ and $i_1 \leq \operatorname{len} f_1$ and $1 \leq i_2$ and $i_2 \leq \operatorname{len} f_1$ holds $(\operatorname{mid}(f_1, i_1, i_2))(\operatorname{len} \operatorname{mid}(f_1, i_1, i_2)) = f_1(i_2)$.

(24) For all f_1, i_1, i_2, j such that $1 \leq i_2$ and $i_2 \leq i_1$ and $i_1 \leq \operatorname{len} f_1$ and $1 \leq j$ and $j \leq (i_1 -' i_2) + 1$ holds $(\operatorname{mid}(f_1, i_1, i_2))(j) = f_1((i_1 -' j) + 1)$.

(25) Let given f_1, i_1, i_2. Suppose $1 \leq i_2$ and $i_2 \leq i_1$ and $i_1 \leq \operatorname{len} f_1$ and $1 \leq j$ and $j \leq (i_1 -' i_2) + 1$. Then $(\operatorname{mid}(f_1, i_1, i_2))(j) = (\operatorname{mid}(f_1, i_2, i_1))((((i_1 - i_2) + 1) - j) + 1)$ and $(((i_1 - i_2) + 1) - j) + 1 = (((i_1 -' i_2) + 1) -' j) + 1$.

(26) Let given f_1, i_1, i_2. Suppose $1 \leq i_1$ and $i_1 \leq i_2$ and $i_2 \leq \operatorname{len} f_1$ and $1 \leq j$ and $j \leq (i_2 -' i_1) + 1$. Then $(\operatorname{mid}(f_1, i_1, i_2))(j) = (\operatorname{mid}(f_1, i_2, i_1))((((i_2 - i_1) + 1) - j) + 1)$ and $(((i_2 - i_1) + 1) - j) + 1 = (((i_2 -' i_1) + 1) -' j) + 1$.

(27) For all f_1, k such that $1 \leq k$ and $k \leq \operatorname{len} f_1$ holds $\operatorname{mid}(f_1, k, k) = \langle (f_1)_k \rangle$ and $\operatorname{len} \operatorname{mid}(f_1, k, k) = 1$.

(28) $\operatorname{mid}(f_1, 0, 0) = f_1 \upharpoonright 1$.

(29) For all f_1, k such that $\operatorname{len} f_1 < k$ holds $\operatorname{mid}(f_1, k, k) = \varepsilon_D$.

(30) For all f_1, i_1, i_2 holds $\operatorname{mid}(f_1, i_1, i_2) = \operatorname{Rev}(\operatorname{mid}(f_1, i_2, i_1))$.

(31) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i_1, i_2, i. If $1 \leq i_1$ and $i_1 < i_2$ and $i_2 \leq \operatorname{len} f$ and $1 \leq i$ and $i < (i_2 -' i_1) + 1$, then $\mathcal{L}(\operatorname{mid}(f, i_1, i_2), i) = \mathcal{L}(f, (i + i_1) -' 1)$.

(32) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i_1, i_2, i. If $1 \leq i_1$ and $i_1 < i_2$ and $i_2 \leq \operatorname{len} f$ and $1 \leq i$ and $i < (i_2 -' i_1) + 1$, then $\mathcal{L}(\operatorname{mid}(f, i_2, i_1), i) = \mathcal{L}(f, i_2 -' i)$.

[1] The propositions (10)–(13) have been removed.

3. Dividing of special circular sequences into parts

Let n be a natural number and let f be a finite sequence. The functor S_Drop(n,f) yields a natural number and is defined as follows:

(Def. 1) $\mathrm{S\_Drop}(n,f) = \begin{cases} n \bmod (\mathrm{len}\, f -' 1), & \text{if } n \bmod (\mathrm{len}\, f -' 1) \neq 0, \\ \mathrm{len}\, f -' 1, & \text{otherwise.} \end{cases}$

We now state three propositions:

(33) For every finite sequence f holds $\mathrm{S\_Drop}(\mathrm{len}\, f -' 1, f) = \mathrm{len}\, f -' 1$.

(34) For every natural number n and for every finite sequence f such that $1 \leq n$ and $n \leq \mathrm{len}\, f -' 1$ holds $\mathrm{S\_Drop}(n,f) = n$.

(35) For every natural number n and for every finite sequence f holds $\mathrm{S\_Drop}(n,f) = \mathrm{S\_Drop}(n + (\mathrm{len}\, f -' 1), f)$ and $\mathrm{S\_Drop}(n,f) = \mathrm{S\_Drop}((\mathrm{len}\, f -' 1) + n, f)$.

Let f be a non constant standard special circular sequence, let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and let i_1, i_2 be natural numbers. We say that g is a right part of f from i_1 to i_2 if and only if the conditions (Def. 2) are satisfied.

(Def. 2) $1 \leq i_1$ and $i_1 + 1 \leq \mathrm{len}\, f$ and $1 \leq i_2$ and $i_2 + 1 \leq \mathrm{len}\, f$ and $g(\mathrm{len}\, g) = f(i_2)$ and $1 \leq \mathrm{len}\, g$ and $\mathrm{len}\, g < \mathrm{len}\, f$ and for every natural number i such that $1 \leq i$ and $i \leq \mathrm{len}\, g$ holds $g(i) = f(\mathrm{S\_Drop}((i_1 + i) -' 1, f))$.

Let f be a non constant standard special circular sequence, let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and let i_1, i_2 be natural numbers. We say that g is a left part of f from i_1 to i_2 if and only if the conditions (Def. 3) are satisfied.

(Def. 3) $1 \leq i_1$ and $i_1 + 1 \leq \mathrm{len}\, f$ and $1 \leq i_2$ and $i_2 + 1 \leq \mathrm{len}\, f$ and $g(\mathrm{len}\, g) = f(i_2)$ and $1 \leq \mathrm{len}\, g$ and $\mathrm{len}\, g < \mathrm{len}\, f$ and for every natural number i such that $1 \leq i$ and $i \leq \mathrm{len}\, g$ holds $g(i) = f(\mathrm{S\_Drop}((\mathrm{len}\, f + i_1) -' i, f))$.

Let f be a non constant standard special circular sequence, let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and let i_1, i_2 be natural numbers. We say that g is a part of f from i_1 to i_2 if and only if:

(Def. 4) g is a right part of f from i_1 to i_2 and a left part of f from i_1 to i_2.

We now state a number of propositions:

(36) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a part of f from i_1 to i_2. Then $1 \leq i_1$ and $i_1 + 1 \leq \mathrm{len}\, f$ and $1 \leq i_2$ and $i_2 + 1 \leq \mathrm{len}\, f$ and $g(\mathrm{len}\, g) = f(i_2)$ and $1 \leq \mathrm{len}\, g$ and $\mathrm{len}\, g < \mathrm{len}\, f$ and for every natural number i such that $1 \leq i$ and $i \leq \mathrm{len}\, g$ holds $g(i) = f(\mathrm{S\_Drop}((i_1 + i) -' 1, f))$ or for every natural number i such that $1 \leq i$ and $i \leq \mathrm{len}\, g$ holds $g(i) = f(\mathrm{S\_Drop}((\mathrm{len}\, f + i_1) -' i, f))$.

(37) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a right part of f from i_1 to i_2 and $i_1 \leq i_2$. Then $\mathrm{len}\, g = (i_2 -' i_1) + 1$ and $g = \mathrm{mid}(f, i_1, i_2)$.

(38) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a right part of f from i_1 to i_2 and $i_1 > i_2$. Then $\mathrm{len}\, g = (\mathrm{len}\, f + i_2) -' i_1$ and $g = (\mathrm{mid}(f, i_1, \mathrm{len}\, f -' 1)) ^\frown (f \restriction i_2)$ and $g = (\mathrm{mid}(f, i_1, \mathrm{len}\, f -' 1)) ^\frown \mathrm{mid}(f, 1, i_2)$.

(39) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a left part of f from i_1 to i_2 and $i_1 \geq i_2$. Then $\mathrm{len}\, g = (i_1 -' i_2) + 1$ and $g = \mathrm{mid}(f, i_1, i_2)$.

(40) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a left part of f from i_1 to i_2 and $i_1 < i_2$. Then $\mathrm{len}\, g = (\mathrm{len}\, f + i_1) -' i_2$ and $g = (\mathrm{mid}(f, i_1, 1)) \frown \mathrm{mid}(f, \mathrm{len}\, f -' 1, i_2)$.

(41) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a right part of f from i_1 to i_2. Then $\mathrm{Rev}(g)$ is a left part of f from i_2 to i_1.

(42) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a left part of f from i_1 to i_2. Then $\mathrm{Rev}(g)$ is a right part of f from i_2 to i_1.

(43) Let f be a non constant standard special circular sequence and i_1, i_2 be natural numbers. If $1 \leq i_1$ and $i_1 \leq i_2$ and $i_2 < \mathrm{len}\, f$, then $\mathrm{mid}(f, i_1, i_2)$ is a right part of f from i_1 to i_2.

(44) Let f be a non constant standard special circular sequence and i_1, i_2 be natural numbers. If $1 \leq i_1$ and $i_1 \leq i_2$ and $i_2 < \mathrm{len}\, f$, then $\mathrm{mid}(f, i_2, i_1)$ is a left part of f from i_2 to i_1.

(45) Let f be a non constant standard special circular sequence and i_1, i_2 be natural numbers. Suppose $1 \leq i_2$ and $i_1 > i_2$ and $i_1 < \mathrm{len}\, f$. Then $(\mathrm{mid}(f, i_1, \mathrm{len}\, f -' 1)) \frown \mathrm{mid}(f, 1, i_2)$ is a right part of f from i_1 to i_2.

(46) Let f be a non constant standard special circular sequence and i_1, i_2 be natural numbers. Suppose $1 \leq i_1$ and $i_1 < i_2$ and $i_2 < \mathrm{len}\, f$. Then $(\mathrm{mid}(f, i_1, 1)) \frown \mathrm{mid}(f, \mathrm{len}\, f -' 1, i_2)$ is a left part of f from i_1 to i_2.

(47) Let h be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i_1, i_2. If $1 \leq i_1$ and $i_1 \leq \mathrm{len}\, h$ and $1 \leq i_2$ and $i_2 \leq \mathrm{len}\, h$, then $\widetilde{\mathcal{L}}(\mathrm{mid}(h, i_1, i_2)) \subseteq \widetilde{\mathcal{L}}(h)$.

(48) Let g be a finite sequence of elements of D. Then g is one-to-one if and only if for all i_1, i_2 such that $1 \leq i_1$ and $i_1 \leq \mathrm{len}\, g$ and $1 \leq i_2$ and $i_2 \leq \mathrm{len}\, g$ and $g(i_1) = g(i_2)$ or $g_{i_1} = g_{i_2}$ holds $i_1 = i_2$.

(49) Let f be a non constant standard special circular sequence and given i_2. If $1 < i_2$ and $i_2 + 1 \leq \mathrm{len}\, f$, then $f \restriction i_2$ is a special sequence.

(50) Let f be a non constant standard special circular sequence and given i_2. If $1 \leq i_2$ and $i_2 + 1 < \mathrm{len}\, f$, then $f_{\downharpoonright i_2}$ is a special sequence.

(51) Let f be a non constant standard special circular sequence and given i_1, i_2. If $1 \leq i_1$ and $i_1 < i_2$ and $i_2 + 1 \leq \mathrm{len}\, f$, then $\mathrm{mid}(f, i_1, i_2)$ is a special sequence.

(52) Let f be a non constant standard special circular sequence and given i_1, i_2. If $1 < i_1$ and $i_1 < i_2$ and $i_2 \leq \mathrm{len}\, f$, then $\mathrm{mid}(f, i_1, i_2)$ is a special sequence.

(53) For all points p_0, p, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_0 \in \mathcal{L}(p, q_1)$ and $p_0 \in \mathcal{L}(p, q_2)$ and $p \neq p_0$ holds $q_1 \in \mathcal{L}(p, q_2)$ or $q_2 \in \mathcal{L}(p, q_1)$.

(54) For every non constant standard special circular sequence f holds $\mathcal{L}(f, 1) \cap \mathcal{L}(f, \mathrm{len}\, f -' 1) = \{f(1)\}$.

(55) Let f be a non constant standard special circular sequence, i_1, i_2 be natural numbers, and g_1, g_2 be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $1 \leq i_1$ and $i_1 < i_2$ and $i_2 < \mathrm{len}\, f$ and $g_1 = \mathrm{mid}(f, i_1, i_2)$ and $g_2 = (\mathrm{mid}(f, i_1, 1)) \frown \mathrm{mid}(f, \mathrm{len}\, f -' 1, i_2)$. Then $\widetilde{\mathcal{L}}(g_1) \cap \widetilde{\mathcal{L}}(g_2) = \{f(i_1), f(i_2)\}$ and $\widetilde{\mathcal{L}}(g_1) \cup \widetilde{\mathcal{L}}(g_2) = \widetilde{\mathcal{L}}(f)$.

(56) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, and i_1, i_2 be natural numbers. Suppose g is a right part of f from i_1 to i_2 and $i_1 < i_2$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining f_{i_1} and f_{i_2}.

(57) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, and i_1, i_2 be natural numbers. Suppose g is a left part of f from i_1 to i_2 and $i_1 > i_2$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining f_{i_1} and f_{i_2}.

(58) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, and i_1, i_2 be natural numbers. Suppose g is a right part of f from i_1 to i_2 and $i_1 \neq i_2$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining f_{i_1} and f_{i_2}.

(59) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, and i_1, i_2 be natural numbers. Suppose g is a left part of f from i_1 to i_2 and $i_1 \neq i_2$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining f_{i_1} and f_{i_2}.

(60) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, and i_1, i_2 be natural numbers. Suppose g is a part of f from i_1 to i_2 and $i_1 \neq i_2$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining f_{i_1} and f_{i_2}.

(61) Let f be a non constant standard special circular sequence, g be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, and i_1, i_2 be natural numbers. Suppose g is a part of f from i_1 to i_2 and $g(1) \neq g(\operatorname{len} g)$. Then $\widetilde{\mathcal{L}}(g)$ is a special polygonal arc joining f_{i_1} and f_{i_2}.

(62) Let f be a non constant standard special circular sequence and i_1, i_2 be natural numbers. Suppose $1 \leq i_1$ and $i_1 + 1 \leq \operatorname{len} f$ and $1 \leq i_2$ and $i_2 + 1 \leq \operatorname{len} f$ and $i_1 \neq i_2$. Then there exist finite sequences g_1, g_2 of elements of $\mathcal{E}^2_{\rm T}$ such that

g_1 is a part of f from i_1 to i_2 and g_2 is a part of f from i_1 to i_2 and $\widetilde{\mathcal{L}}(g_1) \cap \widetilde{\mathcal{L}}(g_2) = \{f(i_1), f(i_2)\}$ and $\widetilde{\mathcal{L}}(g_1) \cup \widetilde{\mathcal{L}}(g_2) = \widetilde{\mathcal{L}}(f)$ and $\widetilde{\mathcal{L}}(g_1)$ is a special polygonal arc joining f_{i_1} and f_{i_2} and $\widetilde{\mathcal{L}}(g_2)$ is a special polygonal arc joining f_{i_1} and f_{i_2} and for every finite sequence g of elements of $\mathcal{E}^2_{\rm T}$ such that g is a part of f from i_1 to i_2 holds $g = g_1$ or $g = g_2$.

In the sequel g_1, g_2 are finite sequences of elements of $\mathcal{E}^2_{\rm T}$.
The following propositions are true:

(63) Let f be a non constant standard special circular sequence and P be a non empty subset of $\mathcal{E}^2_{\rm T}$. If $P = \widetilde{\mathcal{L}}(f)$, then P is a simple closed curve.

(64) Let f be a non constant standard special circular sequence and given g_1, g_2. Suppose g_1 is a right part of f from i_1 to i_2 and g_2 is a right part of f from i_1 to i_2. Then $g_1 = g_2$.

(65) Let f be a non constant standard special circular sequence and given g_1, g_2. Suppose g_1 is a left part of f from i_1 to i_2 and g_2 is a left part of f from i_1 to i_2. Then $g_1 = g_2$.

(66) Let f be a non constant standard special circular sequence and given g_1, g_2. Suppose $i_1 \neq i_2$ and g_1 is a right part of f from i_1 to i_2 and g_2 is a left part of f from i_1 to i_2. Then $g_1(2) \neq g_2(2)$.

(67) Let f be a non constant standard special circular sequence and given g_1, g_2. Suppose $i_1 \neq i_2$ and g_1 is a part of f from i_1 to i_2 and g_2 is a part of f from i_1 to i_2 and $g_1(2) = g_2(2)$. Then $g_1 = g_2$.

Let f be a non constant standard special circular sequence and let i_1, i_2 be natural numbers. Let us assume that $1 \leq i_1$ and $i_1 + 1 \leq \operatorname{len} f$ and $1 \leq i_2$ and $i_2 + 1 \leq \operatorname{len} f$ and $i_1 \neq i_2$. The functor $\operatorname{Lower}(f, i_1, i_2)$ yields a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and is defined by the conditions (Def. 5).

(Def. 5)(i) $\operatorname{Lower}(f, i_1, i_2)$ is a part of f from i_1 to i_2,

(ii) if $(f_{i_1+1})_{\mathbf{1}} < (f_{i_1})_{\mathbf{1}}$ or $(f_{i_1+1})_{\mathbf{2}} < (f_{i_1})_{\mathbf{2}}$, then $(\operatorname{Lower}(f, i_1, i_2))(2) = f(i_1 + 1)$, and

(iii) if $(f_{i_1+1})_{\mathbf{1}} \geq (f_{i_1})_{\mathbf{1}}$ and $(f_{i_1+1})_{\mathbf{2}} \geq (f_{i_1})_{\mathbf{2}}$, then $(\operatorname{Lower}(f, i_1, i_2))(2) = f(\operatorname{S\_Drop}(i_1 -' 1, f))$.

The functor $\operatorname{Upper}(f, i_1, i_2)$ yielding a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ is defined by the conditions (Def. 6).

(Def. 6)(i) Upper(f,i_1,i_2) is a part of f from i_1 to i_2,

(ii) if $(f_{i_1+1})_{\mathbf{1}} > (f_{i_1})_{\mathbf{1}}$ or $(f_{i_1+1})_{\mathbf{2}} > (f_{i_1})_{\mathbf{2}}$, then $(\mathrm{Upper}(f,i_1,i_2))(2) = f(i_1+1)$, and

(iii) if $(f_{i_1+1})_{\mathbf{1}} \leq (f_{i_1})_{\mathbf{1}}$ and $(f_{i_1+1})_{\mathbf{2}} \leq (f_{i_1})_{\mathbf{2}}$, then $(\mathrm{Upper}(f,i_1,i_2))(2) = f(\mathrm{S\_Drop}(i_1 -' 1,f))$.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[4] Czesław Byliński. Some properties of restrictions of finite sequences. *Journal of Formalized Mathematics*, 7, 1995. `http://mizar.org/JFM/Vol7/finseq_5.html`.

[5] Agata Darmochwał. The Euclidean space. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/euclid.html`.

[6] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_\mathrm{T}$. Arcs, line segments and special polygonal arcs. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/topreal1.html`.

[7] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_\mathrm{T}$. Simple closed curves. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/topreal2.html`.

[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[9] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/rfinseq.html`.

[10] Yatsuka Nakamura and Jarosław Kotowicz. Connectedness conditions using polygonal arcs. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/topreal4.html`.

[11] Yatsuka Nakamura and Roman Matuszewski. Reconstructions of special sequences. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/jordan3.html`.

[12] Yatsuka Nakamura and Andrzej Trybulec. Decomposing a Go-Board into cells. *Journal of Formalized Mathematics*, 7, 1995. `http://mizar.org/JFM/Vol7/goboard5.html`.

[13] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/binarith.html`.

[14] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/pre_topc.html`.

[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[16] Andrzej Trybulec. On the decomposition of finite sequences. *Journal of Formalized Mathematics*, 7, 1995. `http://mizar.org/JFM/Vol7/finseq_6.html`.

[17] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_4.html`.

[18] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received May 12, 1997

Published January 2, 2004