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# Bounded Domains and Unbounded Domains

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**Summary.** First, notions of inside components and outside components are introduced for any subset of $n$-dimensional Euclid space. Next, notions of the bounded domain and the unbounded domain are defined using the above components. If the dimension is larger than 1, and if a subset is bounded, a unbounded domain of the subset coincides with an outside component (which is unique) of the subset. For a sphere in $n$-dimensional space, the similar fact is true for a bounded domain. In 2 dimensional space, any rectangle also has such property. We discussed relations between the Jordan property and the concept of boundary, which are necessary to find points in domains near a curve. In the last part, we gave the sufficient criterion for belonging to the left component of some clockwise oriented finite sequences.

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The articles [38], [9], [45], [32], [46], [7], [8], [3], [40], [18], [2], [1], [34], [47], [13], [20], [6], [31], [33], [17], [29], [36], [15], [4], [10], [44], [41], [35], [5], [21], [30], [37], [24], [11], [14], [26], [12], [43], [42], [16], [19], [22], [27], [23], [28], [39], and [25] provide the notation and terminology for this paper.

## 1. Definitions of Bounded Domain and Unbounded Domain

We use the following convention: $m$, $n$ are natural numbers, $r$, $s$ are real numbers, and $x$, $y$ are sets.

We now state several propositions:

(1) If $r \leq 0$, then $|r| = -r$.

(2) For all $n$, $m$ such that $n \leq m$ and $m \leq n+2$ holds $m = n$ or $m = n+1$ or $m = n+2$.

(3) For all $n$, $m$ such that $n \leq m$ and $m \leq n+3$ holds $m = n$ or $m = n+1$ or $m = n+2$ or $m = n+3$.

(4) For all $n$, $m$ such that $n \leq m$ and $m \leq n+4$ holds $m = n$ or $m = n+1$ or $m = n+2$ or $m = n+3$ or $m = n+4$.

(5) For all real numbers $a$, $b$ such that $a \geq 0$ and $b \geq 0$ holds $a+b \geq 0$.

(6) For all real numbers $a$, $b$ such that $a > 0$ and $b \geq 0$ holds $a+b > 0$.

(7) For every finite sequence $f$ such that $\operatorname{rng} f = \{x, y\}$ and $\operatorname{len} f = 2$ holds $f(1) = x$ and $f(2) = y$ or $f(1) = y$ and $f(2) = x$.

(8) Let $f$ be an increasing finite sequence of elements of $\mathbb{R}$. If $\operatorname{rng} f = \{r, s\}$ and $\operatorname{len} f = 2$ and $r \leq s$, then $f(1) = r$ and $f(2) = s$.

In the sequel $p$, $p_1$, $p_2$, $p_3$, $q$, $q_1$, $q_2$ denote points of $\mathcal{E}^n_{\mathrm{T}}$.
One can prove the following propositions:

(9) $(p_1 + p_2) - p_3 = (p_1 - p_3) + p_2$.

(10) $||q|| = |q|$.

(11) $||q_1| - |q_2|| \leq |q_1 - q_2|$.

(12) $||[r]|| = |r|$.

(13) $q - 0_{\mathcal{E}^n_{\mathrm{T}}} = q$ and $0_{\mathcal{E}^n_{\mathrm{T}}} - q = -q$.

(14) For every subset $P$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $P$ is convex holds $P$ is connected.

(15) Let $G$ be a non empty topological space, $P$ be a subset of $G$, $A$ be a subset of $G$, and $Q$ be a subset of $G \restriction A$. If $P = Q$ and $P$ is connected, then $Q$ is connected.

Let us consider $n$ and let $A$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. We say that $A$ is Bounded if and only if:

(Def. 2)[1] There exists a subset $C$ of $\mathcal{E}^n$ such that $C = A$ and $C$ is bounded.

The following proposition is true

(16) For all subsets $A$, $B$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $B$ is Bounded and $A \subseteq B$ holds $A$ is Bounded.

Let us consider $n$, let $A$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and let $B$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. We say that $B$ is inside component of $A$ if and only if:

(Def. 3) $B$ is a component of $A^{\mathrm{c}}$ and Bounded.

Let $M$ be a non empty metric structure. Observe that there exists a subset of $M$ which is bounded.
One can prove the following proposition

(17) Let $A$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$ and $B$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Then $B$ is inside component of $A$ if and only if there exists a subset $C$ of $(\mathcal{E}^n_{\mathrm{T}}) \restriction A^{\mathrm{c}}$ such that $C = B$ and $C$ is a component of $(\mathcal{E}^n_{\mathrm{T}}) \restriction A^{\mathrm{c}}$ and a bounded subset of $\mathcal{E}^n$.

Let us consider $n$, let $A$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and let $B$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. We say that $B$ is outside component of $A$ if and only if:

(Def. 4) $B$ is a component of $A^{\mathrm{c}}$ and $B$ is not Bounded.

The following propositions are true:

(18) Let $A$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$ and $B$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Then $B$ is outside component of $A$ if and only if there exists a subset $C$ of $(\mathcal{E}^n_{\mathrm{T}}) \restriction A^{\mathrm{c}}$ such that $C = B$ and $C$ is a component of $(\mathcal{E}^n_{\mathrm{T}}) \restriction A^{\mathrm{c}}$ and $C$ is not a bounded subset of $\mathcal{E}^n$.

(19) For all subsets $A$, $B$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $B$ is inside component of $A$ holds $B \subseteq A^{\mathrm{c}}$.

(20) For all subsets $A$, $B$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $B$ is outside component of $A$ holds $B \subseteq A^{\mathrm{c}}$.

Let us consider $n$ and let $A$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{BDD}A$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by:

(Def. 5) $\mathrm{BDD}A = \bigcup\{B; B$ ranges over subsets of $\mathcal{E}^n_{\mathrm{T}}$: $B$ is inside component of $A\}$.

Let us consider $n$ and let $A$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{UBD}A$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined as follows:

(Def. 6) $\mathrm{UBD}A = \bigcup\{B; B$ ranges over subsets of $\mathcal{E}^n_{\mathrm{T}}$: $B$ is outside component of $A\}$.

[1] The definition (Def. 1) has been removed.

The following two propositions are true:

(21) $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ is convex.

(22) $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ is connected.

Let us consider $n$. Observe that $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ is connected.
One can prove the following propositions:

(23) $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ is a component of $\mathcal{E}^n_{\mathrm{T}}$.

(24) For every subset $A$ of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{BDD}A$ is a union of components of $(\mathcal{E}^n_{\mathrm{T}})\restriction A^{\mathrm{c}}$.

(25) For every subset $A$ of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{UBD}A$ is a union of components of $(\mathcal{E}^n_{\mathrm{T}})\restriction A^{\mathrm{c}}$.

(26) For every subset $A$ of $\mathcal{E}^n_{\mathrm{T}}$ and for every subset $B$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $B$ is inside component of $A$ holds $B \subseteq \mathrm{BDD}A$.

(27) For every subset $A$ of $\mathcal{E}^n_{\mathrm{T}}$ and for every subset $B$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $B$ is outside component of $A$ holds $B \subseteq \mathrm{UBD}A$.

(28) For every subset $A$ of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{BDD}A$ misses $\mathrm{UBD}A$.

(29) For every subset $A$ of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{BDD}A \subseteq A^{\mathrm{c}}$.

(30) For every subset $A$ of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{UBD}A \subseteq A^{\mathrm{c}}$.

(31) For every subset $A$ of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{BDD}A \cup \mathrm{UBD}A = A^{\mathrm{c}}$.

In the sequel $u$ is a point of $\mathcal{E}^n$.
Next we state two propositions:

(32) Let $G$ be a non empty topological space, $w_1$, $w_2$, $w_3$ be points of $G$, and $h_1$, $h_2$ be maps from $\mathbb{I}$ into $G$. Suppose $h_1$ is continuous and $w_1 = h_1(0)$ and $w_2 = h_1(1)$ and $h_2$ is continuous and $w_2 = h_2(0)$ and $w_3 = h_2(1)$. Then there exists a map $h_3$ from $\mathbb{I}$ into $G$ such that $h_3$ is continuous and $w_1 = h_3(0)$ and $w_3 = h_3(1)$ and $\mathrm{rng}\, h_3 \subseteq \mathrm{rng}\, h_1 \cup \mathrm{rng}\, h_2$.

(33) For every subset $P$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $P = \mathcal{R}^n$ holds $P$ is connected.

Let us consider $n$. The functor $1 * n$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def. 7) $1 * n = n \mapsto (1$ **qua** real number$)$.

Let us consider $n$. Then $1 * n$ is an element of $\mathcal{R}^n$.
Let us consider $n$. The functor $1.\mathrm{REAL}\, n$ yielding a point of $\mathcal{E}^n_{\mathrm{T}}$ is defined as follows:

(Def. 8) $1.\mathrm{REAL}\, n = 1 * n$.

One can prove the following propositions:

(34) $|1 * n| = n \mapsto (1$ **qua** real number$)$.

(35) $|1 * n| = \sqrt{n}$.

(36) $1.\mathrm{REAL}\, 1 = \langle (1$ **qua** real number$) \rangle$.

(37) $|1.\mathrm{REAL}\, n| = \sqrt{n}$.

(38) If $1 \leq n$, then $1 \leq |1.\mathrm{REAL}\, n|$.

(39) For every subset $W$ of $\mathcal{E}^n$ such that $n \geq 1$ and $W = \mathcal{R}^n$ holds $W$ is not bounded.

(40) Let $A$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Then $A$ is Bounded if and only if there exists a real number $r$ such that for every point $q$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $q \in A$ holds $|q| < r$.

(41) If $n \geq 1$, then $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ is not Bounded.

(42) If $n \geq 1$, then $\mathrm{UBD}\,\emptyset_{\mathcal{E}^n_{\mathrm{T}}} = \mathcal{R}^n$.

(43) Let $w_1$, $w_2$, $w_3$ be points of $\mathcal{E}^n_{\mathrm{T}}$, $P$ be a non empty subset of $\mathcal{E}^n_{\mathrm{T}}$, and $h_1$, $h_2$ be maps from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}})\restriction P$. Suppose $h_1$ is continuous and $w_1 = h_1(0)$ and $w_2 = h_1(1)$ and $h_2$ is continuous and $w_2 = h_2(0)$ and $w_3 = h_2(1)$. Then there exists a map $h_3$ from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}})\restriction P$ such that $h_3$ is continuous and $w_1 = h_3(0)$ and $w_3 = h_3(1)$.

(44) Let $P$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$ and $w_1$, $w_2$, $w_3$ be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $w_1 \in P$ and $w_2 \in P$ and $w_3 \in P$ and $\mathcal{L}(w_1,w_2) \subseteq P$ and $\mathcal{L}(w_2,w_3) \subseteq P$. Then there exists a map $h$ from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}})\restriction P$ such that $h$ is continuous and $w_1 = h(0)$ and $w_3 = h(1)$.

(45) Let $P$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$ and $w_1$, $w_2$, $w_3$, $w_4$ be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $w_1 \in P$ and $w_2 \in P$ and $w_3 \in P$ and $w_4 \in P$ and $\mathcal{L}(w_1,w_2) \subseteq P$ and $\mathcal{L}(w_2,w_3) \subseteq P$ and $\mathcal{L}(w_3,w_4) \subseteq P$. Then there exists a map $h$ from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}})\restriction P$ such that $h$ is continuous and $w_1 = h(0)$ and $w_4 = h(1)$.

(46) Let $P$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$ and $w_1$, $w_2$, $w_3$, $w_4$, $w_5$, $w_6$, $w_7$ be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $w_1 \in P$ and $w_2 \in P$ and $w_3 \in P$ and $w_4 \in P$ and $w_5 \in P$ and $w_6 \in P$ and $w_7 \in P$ and $\mathcal{L}(w_1,w_2) \subseteq P$ and $\mathcal{L}(w_2,w_3) \subseteq P$ and $\mathcal{L}(w_3,w_4) \subseteq P$ and $\mathcal{L}(w_4,w_5) \subseteq P$ and $\mathcal{L}(w_5,w_6) \subseteq P$ and $\mathcal{L}(w_6,w_7) \subseteq P$. Then there exists a map $h$ from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}})\restriction P$ such that $h$ is continuous and $w_1 = h(0)$ and $w_7 = h(1)$.

(47) For all points $w_1$, $w_2$ of $\mathcal{E}^n_{\mathrm{T}}$ such that it is not true that there exists a real number $r$ such that $w_1 = r \cdot w_2$ or $w_2 = r \cdot w_1$ holds $0_{\mathcal{E}^n_{\mathrm{T}}} \notin \mathcal{L}(w_1,w_2)$.

(48) Let $w_1$, $w_2$ be points of $\mathcal{E}^n_{\mathrm{T}}$ and $P$ be a subset of $(\mathcal{E}^n)_{\mathrm{top}}$. Suppose $P = \mathcal{L}(w_1,w_2)$ and $0_{\mathcal{E}^n_{\mathrm{T}}} \notin \mathcal{L}(w_1,w_2)$. Then there exists a point $w_0$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $w_0 \in \mathcal{L}(w_1,w_2)$ and $|w_0| > 0$ and $|w_0| = (\mathrm{dist}_{\min}(P))(0_{\mathcal{E}^n_{\mathrm{T}}})$.

(49) Let $a$ be a real number, $Q$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and $w_1$, $w_4$ be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $Q = \{q : |q| > a\}$ and $w_1 \in Q$ and $w_4 \in Q$ and it is not true that there exists a real number $r$ such that $w_1 = r \cdot w_4$ or $w_4 = r \cdot w_1$. Then there exist points $w_2$, $w_3$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $w_2 \in Q$ and $w_3 \in Q$ and $\mathcal{L}(w_1,w_2) \subseteq Q$ and $\mathcal{L}(w_2,w_3) \subseteq Q$ and $\mathcal{L}(w_3,w_4) \subseteq Q$.

(50) Let $a$ be a real number, $Q$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and $w_1$, $w_4$ be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $Q = \mathcal{R}^n \setminus \{q : |q| < a\}$ and $w_1 \in Q$ and $w_4 \in Q$ and it is not true that there exists a real number $r$ such that $w_1 = r \cdot w_4$ or $w_4 = r \cdot w_1$. Then there exist points $w_2$, $w_3$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $w_2 \in Q$ and $w_3 \in Q$ and $\mathcal{L}(w_1,w_2) \subseteq Q$ and $\mathcal{L}(w_2,w_3) \subseteq Q$ and $\mathcal{L}(w_3,w_4) \subseteq Q$.

(52)[2] Every finite sequence $f$ of elements of $\mathbb{R}$ is an element of $\mathcal{R}^{\mathrm{len}\,f}$ and a point of $\mathcal{E}^{\mathrm{len}\,f}_{\mathrm{T}}$.

(53) Let $x$ be an element of $\mathcal{R}^n$, $f$, $g$ be finite sequences of elements of $\mathbb{R}$, and $r$ be a real number. Suppose $f = x$ and $g = r \cdot x$. Then $\mathrm{len}\,f = \mathrm{len}\,g$ and for every natural number $i$ such that $1 \leq i$ and $i \leq \mathrm{len}\,f$ holds $g_i = r \cdot f_i$.

(54) Let $x$ be an element of $\mathcal{R}^n$ and $f$ be a finite sequence. Suppose $x \neq \langle \underbrace{0,\ldots,0}_{n} \rangle$ and $x = f$. Then there exists a natural number $i$ such that $1 \leq i$ and $i \leq n$ and $f(i) \neq 0$.

(55) Let $x$ be an element of $\mathcal{R}^n$. Suppose $n \geq 2$ and $x \neq \langle \underbrace{0,\ldots,0}_{n} \rangle$. Then it is not true that there exists an element $y$ of $\mathcal{R}^n$ and there exists a real number $r$ such that $y = r \cdot x$ or $x = r \cdot y$.

(56) Let $a$ be a real number, $Q$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and $w_1$, $w_7$ be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $n \geq 2$ and $Q = \{q : |q| > a\}$ and $w_1 \in Q$ and $w_7 \in Q$ and there exists a real number $r$ such that $w_1 = r \cdot w_7$ or $w_7 = r \cdot w_1$. Then there exist points $w_2$, $w_3$, $w_4$, $w_5$, $w_6$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $w_2 \in Q$ and $w_3 \in Q$ and $w_4 \in Q$ and $w_5 \in Q$ and $w_6 \in Q$ and $\mathcal{L}(w_1,w_2) \subseteq Q$ and $\mathcal{L}(w_2,w_3) \subseteq Q$ and $\mathcal{L}(w_3,w_4) \subseteq Q$ and $\mathcal{L}(w_4,w_5) \subseteq Q$ and $\mathcal{L}(w_5,w_6) \subseteq Q$ and $\mathcal{L}(w_6,w_7) \subseteq Q$.

[2] The proposition (51) has been removed.

(57) Let $a$ be a real number, $Q$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and $w_1$, $w_7$ be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $n \geq 2$ and $Q = \mathcal{R}^n \setminus \{q : |q| < a\}$ and $w_1 \in Q$ and $w_7 \in Q$ and there exists a real number $r$ such that $w_1 = r \cdot w_7$ or $w_7 = r \cdot w_1$. Then there exist points $w_2$, $w_3$, $w_4$, $w_5$, $w_6$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $w_2 \in Q$ and $w_3 \in Q$ and $w_4 \in Q$ and $w_5 \in Q$ and $w_6 \in Q$ and $\mathcal{L}(w_1, w_2) \subseteq Q$ and $\mathcal{L}(w_2, w_3) \subseteq Q$ and $\mathcal{L}(w_3, w_4) \subseteq Q$ and $\mathcal{L}(w_4, w_5) \subseteq Q$ and $\mathcal{L}(w_5, w_6) \subseteq Q$ and $\mathcal{L}(w_6, w_7) \subseteq Q$.

(58) For every real number $a$ such that $n \geq 1$ holds $\{q : |q| > a\} \neq \emptyset$.

(59) For every real number $a$ and for every subset $P$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $n \geq 2$ and $P = \{q : |q| > a\}$ holds $P$ is connected.

(60) For every real number $a$ such that $n \geq 1$ holds $\mathcal{R}^n \setminus \{q : |q| < a\} \neq \emptyset$.

(61) For every real number $a$ and for every subset $P$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $n \geq 2$ and $P = \mathcal{R}^n \setminus \{q : |q| < a\}$ holds $P$ is connected.

(62) Let $a$ be a real number, $n$ be a natural number, and $P$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. If $n \geq 1$ and $P = \mathcal{R}^n \setminus \{q; q$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $|q| < a\}$, then $P$ is not Bounded.

(63) Let $a$ be a real number and $P$ be a subset of $\mathcal{E}^1_{\mathrm{T}}$. If $P = \{q; q$ ranges over points of $\mathcal{E}^1_{\mathrm{T}}$: $\bigvee_r (q = \langle r \rangle \wedge r > a)\}$, then $P$ is convex.

(64) Let $a$ be a real number and $P$ be a subset of $\mathcal{E}^1_{\mathrm{T}}$. If $P = \{q; q$ ranges over points of $\mathcal{E}^1_{\mathrm{T}}$: $\bigvee_r (q = \langle r \rangle \wedge r < -a)\}$, then $P$ is convex.

(65) Let $a$ be a real number and $P$ be a subset of $\mathcal{E}^1_{\mathrm{T}}$. Suppose $P = \{q; q$ ranges over points of $\mathcal{E}^1_{\mathrm{T}}$: $\bigvee_r (q = \langle r \rangle \wedge r > a)\}$. Then $P$ is connected.

(66) Let $a$ be a real number and $P$ be a subset of $\mathcal{E}^1_{\mathrm{T}}$. Suppose $P = \{q; q$ ranges over points of $\mathcal{E}^1_{\mathrm{T}}$: $\bigvee_r (q = \langle r \rangle \wedge r < -a)\}$. Then $P$ is connected.

(67) Let $W$ be a subset of $\mathcal{E}^1$, $a$ be a real number, and $P$ be a subset of $\mathcal{E}^1_{\mathrm{T}}$. Suppose $W = \{q; q$ ranges over points of $\mathcal{E}^1_{\mathrm{T}}$: $\bigvee_r (q = \langle r \rangle \wedge r > a)\}$ and $P = W$. Then $P$ is connected and $W$ is not bounded.

(68) Let $W$ be a subset of $\mathcal{E}^1$, $a$ be a real number, and $P$ be a subset of $\mathcal{E}^1_{\mathrm{T}}$. Suppose $W = \{q; q$ ranges over points of $\mathcal{E}^1_{\mathrm{T}}$: $\bigvee_r (q = \langle r \rangle \wedge r < -a)\}$ and $P = W$. Then $P$ is connected and $W$ is not bounded.

(69) Let $W$ be a subset of $\mathcal{E}^n$, $a$ be a real number, and $P$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. If $n \geq 2$ and $W = \{q : |q| > a\}$ and $P = W$, then $P$ is connected and $W$ is not bounded.

(70) Let $W$ be a subset of $\mathcal{E}^n$, $a$ be a real number, and $P$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. If $n \geq 2$ and $W = \mathcal{R}^n \setminus \{q : |q| < a\}$ and $P = W$, then $P$ is connected and $W$ is not bounded.

(71) Let $P$, $P_1$ be subsets of $\mathcal{E}^n_{\mathrm{T}}$, $Q$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and $W$ be a subset of $\mathcal{E}^n$. Suppose $P = W$ and $P$ is connected and $W$ is not bounded and $P_1 = \mathrm{Component}(\mathrm{Down}(P, Q^{\mathrm{c}}))$ and $W$ misses $Q$. Then $P_1$ is outside component of $Q$.

(72) Let $A$ be a subset of $\mathcal{E}^n$, $B$ be a non empty subset of $\mathcal{E}^n$, and $C$ be a subset of $\mathcal{E}^n \restriction B$. If $A \subseteq B$ and $A = C$ and $C$ is bounded, then $A$ is bounded.

(73) For every subset $A$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $A$ is compact holds $A$ is Bounded.

(74) For every subset $A$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $1 \leq n$ and $A$ is Bounded holds $A^{\mathrm{c}} \neq \emptyset$.

(75) Let $r$ be a real number. Then there exists a subset $B$ of $\mathcal{E}^n$ such that $B = \{q : |q| < r\}$ and for every subset $A$ of $\mathcal{E}^n$ such that $A = \{q_1 : |q_1| < r\}$ holds $A$ is bounded.

(76) Let $A$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $n \geq 2$ and $A$ is Bounded. Then there exists a subset $B$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $B$ is outside component of $A$ and $B = \mathrm{UBD}\,A$.

(77) For every real number $a$ and for every subset $P$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $P = \{q : |q| < a\}$ holds $P$ is convex.

(78) For every real number $a$ and for every subset $P$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $P = \mathrm{Ball}(u,a)$ holds $P$ is convex.

(79) For every real number $a$ and for every subset $P$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $P = \{q : |q| < a\}$ holds $P$ is connected.

In the sequel $R$ denotes a subset of $\mathcal{E}^n_{\mathrm{T}}$ and $P$ denotes a subset of $\mathcal{E}^n_{\mathrm{T}}$.
Next we state a number of propositions:

(80) Suppose $p \neq q$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$. Then there exists a map $h$ from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$ such that $h$ is continuous and $h(0) = p$ and $h(1) = q$ and $\mathrm{rng}\, h \subseteq \mathrm{Ball}(u,r)$.

(81) Let $f$ be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$. Suppose $f$ is continuous and $f(0) = p_1$ and $f(1) = p_2$ and $p \in \mathrm{Ball}(u,r)$ and $p_2 \in \mathrm{Ball}(u,r)$. Then there exists a map $h$ from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$ such that $h$ is continuous and $h(0) = p_1$ and $h(1) = p$ and $\mathrm{rng}\, h \subseteq \mathrm{rng}\, f \cup \mathrm{Ball}(u,r)$.

(82) Let $f$ be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$. Suppose $f$ is continuous and $\mathrm{rng}\, f \subseteq P$ and $f(0) = p_1$ and $f(1) = p_2$ and $p \in \mathrm{Ball}(u,r)$ and $p_2 \in \mathrm{Ball}(u,r)$ and $\mathrm{Ball}(u,r) \subseteq P$. Then there exists a map $f_1$ from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$ such that $f_1$ is continuous and $\mathrm{rng}\, f_1 \subseteq P$ and $f_1(0) = p_1$ and $f_1(1) = p$.

(83) Let given $p$ and $P$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose that

(i) $R$ is connected and open, and

(ii) $P = \{q : q \neq p \wedge q \in R \wedge \neg \bigvee_{f : \text{map from } \mathbb{I} \text{ into } \mathcal{E}^n_{\mathrm{T}}} (f \text{is continuous} \wedge \mathrm{rng}\, f \subseteq R \wedge f(0) = p \wedge f(1) = q)\}$.

Then $P$ is open.

(84) Let $P$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose that

(i) $R$ is connected and open,

(ii) $p \in R$, and

(iii) $P = \{q : q = p \vee \bigvee_{f : \text{map from } \mathbb{I} \text{ into } \mathcal{E}^n_{\mathrm{T}}} (f \text{is continuous} \wedge \mathrm{rng}\, f \subseteq R \wedge f(0) = p \wedge f(1) = q)\}$.

Then $P$ is open.

(85) Let $R$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $p \in R$ and $P = \{q : q = p \vee \bigvee_{f : \text{map from } \mathbb{I} \text{ into } \mathcal{E}^n_{\mathrm{T}}} (f \text{is continuous} \wedge \mathrm{rng}\, f \subseteq R \wedge f(0) = p \wedge f(1) = q)\}$. Then $P \subseteq R$.

(86) Let $R$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$ and $p$ be a point of $\mathcal{E}^n_{\mathrm{T}}$. Suppose that

(i) $R$ is connected and open,

(ii) $p \in R$, and

(iii) $P = \{q : q = p \vee \bigvee_{f : \text{map from } \mathbb{I} \text{ into } \mathcal{E}^n_{\mathrm{T}}} (f \text{is continuous} \wedge \mathrm{rng}\, f \subseteq R \wedge f(0) = p \wedge f(1) = q)\}$.

Then $R \subseteq P$.

(87) Let $R$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$ and $p$, $q$ be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $R$ is connected and open and $p \in R$ and $q \in R$ and $p \neq q$. Then there exists a map $f$ from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$ such that $f$ is continuous and $\mathrm{rng}\, f \subseteq R$ and $f(0) = p$ and $f(1) = q$.

(88) For every subset $A$ of $\mathcal{E}^n_{\mathrm{T}}$ and for every real number $a$ such that $A = \{q : |q| = a\}$ holds $A^{\mathrm{c}}$ is open and $A$ is closed.

(89) For every non empty subset $B$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $B$ is open holds $(\mathcal{E}^n_{\mathrm{T}}) \restriction B$ is locally connected.

(90) Let $B$ be a non empty subset of $\mathcal{E}^n_{\mathrm{T}}$, $A$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and $a$ be a real number. If $A = \{q : |q| = a\}$ and $A^{\mathrm{c}} = B$, then $(\mathcal{E}^n_{\mathrm{T}}) \restriction B$ is locally connected.

(91) For every map $f$ from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ such that for every $q$ holds $f(q) = |q|$ holds $f$ is continuous.

(92) There exists a map $f$ from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ such that for every $q$ holds $f(q) = |q|$ and $f$ is continuous.

Let $X$, $Y$ be non empty 1-sorted structures, let $f$ be a map from $X$ into $Y$, and let $x$ be a set. Let us assume that $x$ is a point of $X$. The functor $\pi_x f$ yields a point of $Y$ and is defined as follows:

(Def. 10)[3] $\pi_x f = f(x)$.

The following four propositions are true:

(93) Let $g$ be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$. Suppose $g$ is continuous. Then there exists a map $f$ from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ such that for every point $t$ of $\mathbb{I}$ holds $f(t) = |g(t)|$ and $f$ is continuous.

(94) Let $g$ be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$ and $a$ be a real number. Suppose $g$ is continuous and $|\pi_0 g| \leq a$ and $a \leq |\pi_1 g|$. Then there exists a point $s$ of $\mathbb{I}$ such that $|\pi_s g| = a$.

(95) If $q = \langle r \rangle$, then $|q| = |r|$.

(96) Let $A$ be a subset of $\mathcal{E}^n_{\mathrm{T}}$ and $a$ be a real number. Suppose $n \geq 1$ and $a > 0$ and $A = \{q : |q| = a\}$. Then there exists a subset $B$ of $\mathcal{E}^n_{\mathrm{T}}$ such that $B$ is inside component of $A$ and $B = \mathrm{BDD}\, A$.

## 2. Bounded and Unbounded Domains of Rectangles

In the sequel $D$ denotes a non vertical non horizontal non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$.

The following propositions are true:

(97) len the Go-board of $\mathrm{SpStSeq}\, D = 2$ and width the Go-board of $\mathrm{SpStSeq}\, D = 2$ and $(\mathrm{SpStSeq}\, D)_1 =$ the Go-board of $\mathrm{SpStSeq}\, D \circ (1,2)$ and $(\mathrm{SpStSeq}\, D)_2 =$ the Go-board of $\mathrm{SpStSeq}\, D \circ (2,2)$ and $(\mathrm{SpStSeq}\, D)_3 =$ the Go-board of $\mathrm{SpStSeq}\, D \circ (2,1)$ and $(\mathrm{SpStSeq}\, D)_4 =$ the Go-board of $\mathrm{SpStSeq}\, D \circ (1,1)$ and $(\mathrm{SpStSeq}\, D)_5 =$ the Go-board of $\mathrm{SpStSeq}\, D \circ (1,2)$.

(98) $\mathrm{LeftComp}(\mathrm{SpStSeq}\, D)$ is non Bounded.

(99) $\mathrm{LeftComp}(\mathrm{SpStSeq}\, D) \subseteq \mathrm{UBD}\, \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$.

(100) Let $G$ be a topological space and $A$, $B$, $C$ be subsets of $G$. Suppose $A$ is a component of $G$ and $B$ is a component of $G$ and $C$ is connected and $A$ meets $C$ and $B$ meets $C$. Then $A = B$.

(101) For every subset $B$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $B$ is a component of $(\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D))^{\mathrm{c}}$ and $B$ is not Bounded holds $B = \mathrm{LeftComp}(\mathrm{SpStSeq}\, D)$.

(102) $\mathrm{RightComp}(\mathrm{SpStSeq}\, D) \subseteq \mathrm{BDD}\, \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$ and $\mathrm{RightComp}(\mathrm{SpStSeq}\, D)$ is Bounded.

(103) $\mathrm{LeftComp}(\mathrm{SpStSeq}\, D) = \mathrm{UBD}\, \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$ and $\mathrm{RightComp}(\mathrm{SpStSeq}\, D) = \mathrm{BDD}\, \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$.

(104) $\mathrm{UBD}\, \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) \neq \emptyset$ and $\mathrm{UBD}\, \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$ is outside component of $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$ and $\mathrm{BDD}\, \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D) \neq \emptyset$ and $\mathrm{BDD}\, \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$ is inside component of $\widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$.

[3] The definition (Def. 9) has been removed.

## 3. JORDAN PROPERTY AND BOUNDARY PROPERTY

We now state several propositions:

(105) Let $G$ be a non empty topological space and $A$ be a subset of $G$. Suppose $A^{\rm c} \neq \emptyset$. Then $A$ is boundary if and only if for every set $x$ and for every subset $V$ of $G$ such that $x \in A$ and $x \in V$ and $V$ is open there exists a subset $B$ of $G$ such that $B$ is a component of $A^{\rm c}$ and $V$ meets $B$.

(106) Let $A$ be a subset of $\mathcal{E}^2_{\rm T}$. Suppose $A^{\rm c} \neq \emptyset$. Then $A$ is boundary and Jordan if and only if there exist subsets $A_1, A_2$ of $\mathcal{E}^2_{\rm T}$ such that $A^{\rm c} = A_1 \cup A_2$ and $A_1$ misses $A_2$ and $\overline{A_1} \setminus A_1 = \overline{A_2} \setminus A_2$ and $A = \overline{A_1} \setminus A_1$ and for all subsets $C_1$, $C_2$ of $(\mathcal{E}^2_{\rm T}) \restriction A^{\rm c}$ such that $C_1 = A_1$ and $C_2 = A_2$ holds $C_1$ is a component of $(\mathcal{E}^2_{\rm T}) \restriction A^{\rm c}$ and $C_2$ is a component of $(\mathcal{E}^2_{\rm T}) \restriction A^{\rm c}$.

(107) For every point $p$ of $\mathcal{E}^n_{\rm T}$ and for every subset $P$ of $\mathcal{E}^n_{\rm T}$ such that $n \geq 1$ and $P = \{p\}$ holds $P$ is boundary.

(108) For all points $p$, $q$ of $\mathcal{E}^2_{\rm T}$ and for every $r$ such that $p_{\mathbf{1}} = q_{\mathbf{2}}$ and $-p_{\mathbf{2}} = q_{\mathbf{1}}$ and $p = r \cdot q$ holds $p_{\mathbf{1}} = 0$ and $p_{\mathbf{2}} = 0$ and $p = 0_{\mathcal{E}^2_{\rm T}}$.

(109) For all points $q_1$, $q_2$ of $\mathcal{E}^2_{\rm T}$ holds $\mathcal{L}(q_1, q_2)$ is boundary.

Let $q_1$, $q_2$ be points of $\mathcal{E}^2_{\rm T}$. One can verify that $\mathcal{L}(q_1, q_2)$ is boundary.
One can prove the following proposition

(110) For every finite sequence $f$ of elements of $\mathcal{E}^2_{\rm T}$ holds $\widetilde{\mathcal{L}}(f)$ is boundary.

Let $f$ be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Note that $\widetilde{\mathcal{L}}(f)$ is boundary.
Next we state several propositions:

(111) For every point $e_1$ of $\mathcal{E}^n$ and for all points $p$, $q$ of $\mathcal{E}^n_{\rm T}$ such that $p = e_1$ and $q \in \mathrm{Ball}(e_1, r)$ holds $|p - q| < r$ and $|q - p| < r$.

(112) Let $a$ be a real number and $p$ be a point of $\mathcal{E}^2_{\rm T}$. Suppose $a > 0$ and $p \in \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$. Then there exists a point $q$ of $\mathcal{E}^2_{\rm T}$ such that $q \in \mathrm{UBD}\, \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$ and $|p - q| < a$.

(113) $\mathcal{R}^0 = \{0_{\mathcal{E}^0_{\rm T}}\}$.

(114) For every subset $A$ of $\mathcal{E}^n_{\rm T}$ such that $A$ is Bounded holds $\mathrm{BDD}\, A$ is Bounded.

(115) Let $G$ be a non empty topological space and $A$, $B$, $C$, $D$ be subsets of $G$. Suppose $A$ is a component of $G$ and $B$ is a component of $G$ and $C$ is a component of $G$ and $A \cup B =$ the carrier of $G$ and $C$ misses $A$. Then $C = B$.

(116) For every subset $A$ of $\mathcal{E}^2_{\rm T}$ such that $A$ is Bounded and Jordan holds $\mathrm{BDD}\, A$ is inside component of $A$.

(117) Let $a$ be a real number and $p$ be a point of $\mathcal{E}^2_{\rm T}$. Suppose $a > 0$ and $p \in \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$. Then there exists a point $q$ of $\mathcal{E}^2_{\rm T}$ such that $q \in \mathrm{BDD}\, \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$ and $|p - q| < a$.

## 4. POINTS IN LEFTCOMP

In the sequel $f$ is a clockwise oriented non constant standard special circular sequence.
One can prove the following propositions:

(118) For every point $p$ of $\mathcal{E}^2_{\rm T}$ such that $f_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f))$ and $p_{\mathbf{1}} < \text{W-bound}(\widetilde{\mathcal{L}}(f))$ holds $p \in \mathrm{LeftComp}(f)$.

(119) For every point $p$ of $\mathcal{E}^2_{\rm T}$ such that $f_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f))$ and $p_{\mathbf{1}} > \text{E-bound}(\widetilde{\mathcal{L}}(f))$ holds $p \in \mathrm{LeftComp}(f)$.

(120) For every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $f_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f))$ and $p_{\mathbf{2}} < \text{S-bound}(\widetilde{\mathcal{L}}(f))$ holds $p \in \mathrm{LeftComp}(f)$.

(121) For every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $f_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f))$ and $p_{\mathbf{2}} > \text{N-bound}(\widetilde{\mathcal{L}}(f))$ holds $p \in \mathrm{LeftComp}(f)$.

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