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General Fashoda Meet Theorem for Unit Circle and Square

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Summary. Here we will prove Fashoda meet theorem for the unit circle and for a square, when 4 points on the boundary are ordered cyclically. Also, the concepts of general rectangle and general circle are defined.

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The articles [1], [9], [22], [27], [8], [4], [5], [26], [2], [10], [3], [7], [14], [24], [20], [19], [17], [18], [12], [25], [15], [16], [23], [21], [11], [6], and [13] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(2)[1] For all real numbers a, b, r such that $0 \leq r$ and $r \leq 1$ and $a \leq b$ holds $a \leq (1-r)\cdot a + r \cdot b$ and $(1-r)\cdot a + r\cdot b \leq b$.

(3) For all real numbers a, b such that $a \geq 0$ and $b > 0$ or $a > 0$ and $b \geq 0$ holds $a + b > 0$.

(4) For all real numbers a, b such that $-1 \leq a$ and $a \leq 1$ and $-1 \leq b$ and $b \leq 1$ holds $a^{\mathbf{2}} \cdot b^{\mathbf{2}} \leq 1$.

(5) For all real numbers a, b such that $a \geq 0$ and $b \geq 0$ holds $a \cdot \sqrt{b} = \sqrt{a^{\mathbf{2}} \cdot b}$.

(6) For all real numbers a, b such that $-1 \leq a$ and $a \leq 1$ and $-1 \leq b$ and $b \leq 1$ holds $(-b)\cdot \sqrt{1+a^{\mathbf{2}}} \leq \sqrt{1+b^{\mathbf{2}}}$ and $-\sqrt{1+b^{\mathbf{2}}} \leq b \cdot \sqrt{1+a^{\mathbf{2}}}$.

(7) For all real numbers a, b such that $-1 \leq a$ and $a \leq 1$ and $-1 \leq b$ and $b \leq 1$ holds $b \cdot \sqrt{1+a^{\mathbf{2}}} \leq \sqrt{1+b^{\mathbf{2}}}$.

(8) For all real numbers a, b such that $a \geq b$ holds $a \cdot \sqrt{1+b^{\mathbf{2}}} \geq b \cdot \sqrt{1+a^{\mathbf{2}}}$.

(9) Let a, c, d be real numbers and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $c \leq d$ and $p \in \mathcal{L}([a,c],[a,d])$, then $p_{\mathbf{1}} = a$ and $c \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq d$.

(10) For all real numbers a, c, d and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $c < d$ and $p_{\mathbf{1}} = a$ and $c \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq d$ holds $p \in \mathcal{L}([a,c],[a,d])$.

[1] The proposition (1) has been removed.

(11) Let a, b, d be real numbers and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $a \leq b$ and $p \in \mathcal{L}([a,d],[b,d])$, then $p_{\mathbf{2}} = d$ and $a \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq b$.

(12) For all real numbers a, b and for every subset B of $\mathbb{I}$ such that $B = [a,b]$ holds B is closed.

(13) Let X be a topological structure, Y, Z be non empty topological structures, f be a map from X into Y, and g be a map from X into Z. Then $\mathrm{dom} f = \mathrm{dom} g$ and $\mathrm{dom} f =$ the carrier of X and $\mathrm{dom} f = \Omega_X$.

(14) Let X be a non empty topological space and B be a non empty subset of X. Then there exists a map f from $X \restriction B$ into X such that for every point p of $X \restriction B$ holds $f(p) = p$ and f is continuous.

(15) Let X be a non empty topological space, f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$, and a be a real number. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = r_1 - a$ and g is continuous.

(16) Let X be a non empty topological space, f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$, and a be a real number. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = a - r_1$ and g is continuous.

(17) Let X be a non empty topological space, n be a natural number, p be a point of $\mathcal{E}^n_{\mathrm{T}}$, and f be a map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f is continuous. Then there exists a map g from X into $\mathcal{E}^n_{\mathrm{T}}$ such that for every point r of X holds $g(r) = f(r) \cdot p$ and g is continuous.

(18) $\mathrm{SqCirc}([-1,0]) = [-1,0]$.

(19) For every compact non empty subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ holds $\mathrm{SqCirc}([-1,0]) = \mathrm{W}_{\mathrm{min}}(P)$.

(20) Let X be a non empty topological space, n be a natural number, and g_1, g_2 be maps from X into $\mathcal{E}^n_{\mathrm{T}}$. Suppose g_1 is continuous and g_2 is continuous. Then there exists a map g from X into $\mathcal{E}^n_{\mathrm{T}}$ such that for every point r of X holds $g(r) = g_1(r) + g_2(r)$ and g is continuous.

(21) Let X be a non empty topological space, n be a natural number, p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$, and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous. Then there exists a map g from X into $\mathcal{E}^n_{\mathrm{T}}$ such that for every point r of X holds $g(r) = f_1(r) \cdot p_1 + f_2(r) \cdot p_2$ and g is continuous.

(22) For every function f and for every set A such that f is one-to-one and $A \subseteq \mathrm{dom} f$ holds $(f^{-1})^{\circ} f^{\circ} A = A$.

2. General Fashoda Theorem for Unit Circle

In the sequel p, p_1, p_2, p_3, q, q_1, q_2 denote points of $\mathcal{E}^2_{\mathrm{T}}$.

The following propositions are true:

(23) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, C_0, K_1, K_2, K_3, K_4 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leq 1\}$ and $K_1 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_1| = 1 \wedge (q_1)_{\mathbf{2}} \leq (q_1)_{\mathbf{1}} \wedge (q_1)_{\mathbf{2}} \geq -(q_1)_{\mathbf{1}}\}$ and $K_2 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_2| = 1 \wedge (q_2)_{\mathbf{2}} \geq (q_2)_{\mathbf{1}} \wedge (q_2)_{\mathbf{2}} \leq -(q_2)_{\mathbf{1}}\}$ and $K_3 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_3| = 1 \wedge (q_3)_{\mathbf{2}} \geq (q_3)_{\mathbf{1}} \wedge (q_3)_{\mathbf{2}} \geq -(q_3)_{\mathbf{1}}\}$ and $K_4 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_4| = 1 \wedge (q_4)_{\mathbf{2}} \leq (q_4)_{\mathbf{1}} \wedge (q_4)_{\mathbf{2}} \leq -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_1$ and $f(I) \in K_2$ and $g(O) \in K_3$ and $g(I) \in K_4$ and $\mathrm{rng} f \subseteq C_0$ and $\mathrm{rng} g \subseteq C_0$. Then $\mathrm{rng} f$ meets $\mathrm{rng} g$.

(24) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, C_0, K_1, K_2, K_3, K_4 be subsets of $\mathcal{E}^2_{\rm T}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leq 1\}$ and $K_1 = \{q_1;q_1$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_1| = 1 \wedge (q_1)_{\mathbf{2}} \leq (q_1)_{\mathbf{1}} \wedge (q_1)_{\mathbf{2}} \geq -(q_1)_{\mathbf{1}}\}$ and $K_2 = \{q_2;q_2$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_2| = 1 \wedge (q_2)_{\mathbf{2}} \geq (q_2)_{\mathbf{1}} \wedge (q_2)_{\mathbf{2}} \leq -(q_2)_{\mathbf{1}}\}$ and $K_3 = \{q_3;q_3$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_3| = 1 \wedge (q_3)_{\mathbf{2}} \geq (q_3)_{\mathbf{1}} \wedge (q_3)_{\mathbf{2}} \geq -(q_3)_{\mathbf{1}}\}$ and $K_4 = \{q_4;q_4$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_4| = 1 \wedge (q_4)_{\mathbf{2}} \leq (q_4)_{\mathbf{1}} \wedge (q_4)_{\mathbf{2}} \leq -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_1$ and $f(I) \in K_2$ and $g(O) \in K_4$ and $g(I) \in K_3$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(25) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$, and C_0 be a subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p;p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p_8;p_8$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p_8| \leq 1\}$ and $f(0) = p_3$ and $f(1) = p_1$ and $g(0) = p_2$ and $g(1) = p_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(26) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$, and C_0 be a subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p;p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p_8;p_8$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p_8| \leq 1\}$ and $f(0) = p_3$ and $f(1) = p_1$ and $g(0) = p_4$ and $g(1) = p_2$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(27) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$, and C_0 be a subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p;p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$ and p_1, p_2, p_3, p_4 are in this order on P. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p_8;p_8$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p_8| \leq 1\}$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

3. General Rectangles and Circles

Let a, b, c, d be real numbers. The functor Rectangle(a,b,c,d) yields a subset of $\mathcal{E}^2_{\rm T}$ and is defined by the condition (Def. 1).

(Def. 1) $\operatorname{Rectangle}(a,b,c,d) = \{p : p_{\mathbf{1}} = a \wedge c \leq p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leq d \vee p_{\mathbf{2}} = d \wedge a \leq p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leq b \vee p_{\mathbf{1}} = b \wedge c \leq p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leq d \vee p_{\mathbf{2}} = c \wedge a \leq p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leq b\}$.

The following proposition is true

(28) Let a, b, c, d be real numbers and p be a point of $\mathcal{E}^2_{\rm T}$. If $a \leq b$ and $c \leq d$ and $p \in \operatorname{Rectangle}(a,b,c,d)$, then $a \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq b$ and $c \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq d$.

Let a, b, c, d be real numbers. The functor InsideOfRectangle(a,b,c,d) yielding a subset of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 2) $\operatorname{InsideOfRectangle}(a,b,c,d) = \{p : a < p_{\mathbf{1}} \wedge p_{\mathbf{1}} < b \wedge c < p_{\mathbf{2}} \wedge p_{\mathbf{2}} < d\}$.

Let a, b, c, d be real numbers. The functor ClosedInsideOfRectangle(a,b,c,d) yields a subset of $\mathcal{E}^2_{\rm T}$ and is defined by:

(Def. 3) $\operatorname{ClosedInsideOfRectangle}(a,b,c,d) = \{p : a \leq p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leq b \wedge c \leq p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leq d\}$.

Let a, b, c, d be real numbers. The functor OutsideOfRectangle(a,b,c,d) yields a subset of $\mathcal{E}^2_{\rm T}$ and is defined by:

(Def. 4) $\operatorname{OutsideOfRectangle}(a,b,c,d) = \{p : a \not\leq p_{\mathbf{1}} \vee p_{\mathbf{1}} \not\leq b \vee c \not\leq p_{\mathbf{2}} \vee p_{\mathbf{2}} \not\leq d\}$.

Let a, b, c, d be real numbers. The functor ClosedOutsideOfRectangle(a,b,c,d) yielding a subset of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 5) ClosedOutsideOfRectangle$(a,b,c,d) = \{p : a \not< p_{\mathbf{1}} \lor p_{\mathbf{1}} \not< b \lor c \not< p_{\mathbf{2}} \lor p_{\mathbf{2}} \not< d\}$.

We now state four propositions:

(29) Let a, b, r be real numbers and K_5, C_1 be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $r \geq 0$ and $K_5 = \{q : |q| = 1\}$ and $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p_2 - [a,b]| = r\}$. Then $({\rm AffineMap}(r,a,r,b))^\circ K_5 = C_1$.

(30) Let P, Q be subsets of $\mathcal{E}^2_{\rm T}$. Suppose there exists a map from $\mathcal{E}^2_{\rm T} \restriction P$ into $\mathcal{E}^2_{\rm T} \restriction Q$ which is a homeomorphism and P is a simple closed curve. Then Q is a simple closed curve.

(31) For every subset P of $\mathcal{E}^2_{\rm T}$ such that P satisfies conditions of simple closed curve holds P is compact.

(32) Let a, b, r be real numbers and C_1 be a subset of $\mathcal{E}^2_{\rm T}$. Suppose $r > 0$ and $C_1 = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p - [a,b]| = r\}$. Then C_1 is a simple closed curve.

Let a, b, r be real numbers. Let us assume that $r > 0$. The functor Circle(a,b,r) yielding a compact non empty subset of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 6) Circle$(a,b,r) = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p - [a,b]| = r\}$.

Let a, b, r be real numbers. The functor InsideOfCircle(a,b,r) yields a subset of $\mathcal{E}^2_{\rm T}$ and is defined by:

(Def. 7) InsideOfCircle$(a,b,r) = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p - [a,b]| < r\}$.

Let a, b, r be real numbers. The functor ClosedInsideOfCircle(a,b,r) yields a subset of $\mathcal{E}^2_{\rm T}$ and is defined as follows:

(Def. 8) ClosedInsideOfCircle$(a,b,r) = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p - [a,b]| \leq r\}$.

Let a, b, r be real numbers. The functor OutsideOfCircle(a,b,r) yielding a subset of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 9) OutsideOfCircle$(a,b,r) = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p - [a,b]| > r\}$.

Let a, b, r be real numbers. The functor ClosedOutsideOfCircle(a,b,r) yielding a subset of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 10) ClosedOutsideOfCircle$(a,b,r) = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p - [a,b]| \geq r\}$.

One can prove the following propositions:

(33) Let r be a real number. Then InsideOfCircle$(0,0,r) = \{p : |p| < r\}$ and if $r > 0$, then Circle$(0,0,r) = \{p_2 : |p_2| = r\}$ and OutsideOfCircle$(0,0,r) = \{p_3 : |p_3| > r\}$ and ClosedInsideOfCircle$(0,0,r) = \{q : |q| \leq r\}$ and ClosedOutsideOfCircle$(0,0,r) = \{q_2 : |q_2| \geq r\}$.

(34) Let K_5, C_1 be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $K_5 = \{p : -1 < p_{\mathbf{1}} \land p_{\mathbf{1}} < 1 \land -1 < p_{\mathbf{2}} \land p_{\mathbf{2}} < 1\}$ and $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p_2| < 1\}$. Then SqCirc$^\circ K_5 = C_1$.

(35) Let K_5, C_1 be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $K_5 = \{p : -1 \not\leq p_{\mathbf{1}} \lor p_{\mathbf{1}} \not\leq 1 \lor -1 \not\leq p_{\mathbf{2}} \lor p_{\mathbf{2}} \not\leq 1\}$ and $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p_2| > 1\}$. Then SqCirc$^\circ K_5 = C_1$.

(36) Let K_5, C_1 be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $K_5 = \{p : -1 \leq p_{\mathbf{1}} \land p_{\mathbf{1}} \leq 1 \land -1 \leq p_{\mathbf{2}} \land p_{\mathbf{2}} \leq 1\}$ and $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p_2| \leq 1\}$. Then SqCirc$^\circ K_5 = C_1$.

(37) Let K_5, C_1 be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $K_5 = \{p : -1 \not< p_{\mathbf{1}} \lor p_{\mathbf{1}} \not< 1 \lor -1 \not< p_{\mathbf{2}} \lor p_{\mathbf{2}} \not< 1\}$ and $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p_2| \geq 1\}$. Then SqCirc$^\circ K_5 = C_1$.

(38) Let P_0, P_1, P_2, P_{11}, K_0, K_6, K_7, K_{11} be subsets of $\mathcal{E}^2_{\rm T}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. Suppose that $P = \text{Circle}(0,0,1)$ and $P_0 = \text{InsideOfCircle}(0,0,1)$ and $P_1 = \text{OutsideOfCircle}(0,0,1)$ and $P_2 = \text{ClosedInsideOfCircle}(0,0,1)$ and $P_{11} = \text{ClosedOutsideOfCircle}(0,0,1)$ and $K = \text{Rectangle}(-1,1,-1,1)$ and $K_0 = \text{InsideOfRectangle}(-1,1,-1,1)$ and $K_6 = \text{OutsideOfRectangle}(-1,1,-1,1)$ and $K_7 = \text{ClosedInsideOfRectangle}(-1,1,-1,1)$ and $K_{11} = \text{ClosedOutsideOfRectangle}(-1,1,-1,1)$ and $f = \text{SqCirc}$. Then $f^\circ K = P$ and $(f^{-1})^\circ P = K$ and $f^\circ K_0 = P_0$ and $(f^{-1})^\circ P_0 = K_0$ and $f^\circ K_6 = P_1$ and $(f^{-1})^\circ P_1 = K_6$ and $f^\circ K_7 = P_2$ and $f^\circ K_{11} = P_{11}$ and $(f^{-1})^\circ P_2 = K_7$ and $(f^{-1})^\circ P_{11} = K_{11}$.

4. ORDER OF POINTS ON RECTANGLE

One can prove the following propositions:

(39) Let a, b, c, d be real numbers. Suppose $a \leq b$ and $c \leq d$. Then

(i) $\mathcal{L}([a,c],[a,d]) = \{p_1 : (p_1)_{\mathbf{1}} = a \wedge (p_1)_{\mathbf{2}} \leq d \wedge (p_1)_{\mathbf{2}} \geq c\}$,

(ii) $\mathcal{L}([a,d],[b,d]) = \{p_2 : (p_2)_{\mathbf{1}} \leq b \wedge (p_2)_{\mathbf{1}} \geq a \wedge (p_2)_{\mathbf{2}} = d\}$,

(iii) $\mathcal{L}([a,c],[b,c]) = \{q_1 : (q_1)_{\mathbf{1}} \leq b \wedge (q_1)_{\mathbf{1}} \geq a \wedge (q_1)_{\mathbf{2}} = c\}$, and

(iv) $\mathcal{L}([b,c],[b,d]) = \{q_2 : (q_2)_{\mathbf{1}} = b \wedge (q_2)_{\mathbf{2}} \leq d \wedge (q_2)_{\mathbf{2}} \geq c\}$.

(40) Let a, b, c, d be real numbers. Suppose $a \leq b$ and $c \leq d$. Then $\{p : p_{\mathbf{1}} = a \wedge c \leq p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leq d \vee p_{\mathbf{2}} = d \wedge a \leq p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leq b \vee p_{\mathbf{1}} = b \wedge c \leq p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leq d \vee p_{\mathbf{2}} = c \wedge a \leq p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leq b\} = \mathcal{L}([a,c],[a,d]) \cup \mathcal{L}([a,d],[b,d]) \cup (\mathcal{L}([a,c],[b,c]) \cup \mathcal{L}([b,c],[b,d]))$.

(41) For all real numbers a, b, c, d such that $a \leq b$ and $c \leq d$ holds $\mathcal{L}([a,c],[a,d]) \cap \mathcal{L}([a,c],[b,c]) = \{[a,c]\}$.

(42) For all real numbers a, b, c, d such that $a \leq b$ and $c \leq d$ holds $\mathcal{L}([a,c],[b,c]) \cap \mathcal{L}([b,c],[b,d]) = \{[b,c]\}$.

(43) For all real numbers a, b, c, d such that $a \leq b$ and $c \leq d$ holds $\mathcal{L}([a,d],[b,d]) \cap \mathcal{L}([b,c],[b,d]) = \{[b,d]\}$.

(44) For all real numbers a, b, c, d such that $a \leq b$ and $c \leq d$ holds $\mathcal{L}([a,c],[a,d]) \cap \mathcal{L}([a,d],[b,d]) = \{[a,d]\}$.

(45) $\{q : -1 = q_{\mathbf{1}} \wedge -1 \leq q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leq 1 \vee q_{\mathbf{1}} = 1 \wedge -1 \leq q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leq 1 \vee -1 = q_{\mathbf{2}} \wedge -1 \leq q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leq 1 \vee 1 = q_{\mathbf{2}} \wedge -1 \leq q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leq 1\} = \{p : p_{\mathbf{1}} = -1 \wedge -1 \leq p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leq 1 \vee p_{\mathbf{2}} = 1 \wedge -1 \leq p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leq 1 \vee p_{\mathbf{1}} = 1 \wedge -1 \leq p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leq 1 \vee p_{\mathbf{2}} = -1 \wedge -1 \leq p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leq 1\}$.

(46) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \text{Rectangle}(a,b,c,d)$ and $a \leq b$ and $c \leq d$, then $\text{W-bound}(K) = a$.

(47) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \text{Rectangle}(a,b,c,d)$ and $a \leq b$ and $c \leq d$, then $\text{N-bound}(K) = d$.

(48) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \text{Rectangle}(a,b,c,d)$ and $a \leq b$ and $c \leq d$, then $\text{E-bound}(K) = b$.

(49) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \text{Rectangle}(a,b,c,d)$ and $a \leq b$ and $c \leq d$, then $\text{S-bound}(K) = c$.

(50) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \text{Rectangle}(a,b,c,d)$ and $a \leq b$ and $c \leq d$, then $\text{NW-corner}(K) = [a,d]$.

(51) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \text{Rectangle}(a,b,c,d)$ and $a \leq b$ and $c \leq d$, then $\text{NE-corner}(K) = [b,d]$.

(52) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leq b$ and $c \leq d$, then $\text{SW-corner}(K) = [a,c]$.

(53) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leq b$ and $c \leq d$, then $\text{SE-corner}(K) = [b,c]$.

(54) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leq b$ and $c \leq d$, then $\mathrm{W}_{\mathrm{most}}(K) = \mathcal{L}([a,c],[a,d])$.

(55) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leq b$ and $c \leq d$, then $\mathrm{E}_{\mathrm{most}}(K) = \mathcal{L}([b,c],[b,d])$.

(56) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leq b$ and $c \leq d$, then $\mathrm{W}_{\mathrm{min}}(K) = [a,c]$ and $\mathrm{E}_{\mathrm{max}}(K) = [b,d]$.

(57) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$. Then $\mathcal{L}([a,c],[a,d]) \cup \mathcal{L}([a,d],[b,d])$ is an arc from $\mathrm{W}_{\mathrm{min}}(K)$ to $\mathrm{E}_{\mathrm{max}}(K)$ and $\mathcal{L}([a,c],[b,c]) \cup \mathcal{L}([b,c],[b,d])$ is an arc from $\mathrm{E}_{\mathrm{max}}(K)$ to $\mathrm{W}_{\mathrm{min}}(K)$.

(58) Let P, P_1, P_3 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, f_1, f_2 be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, and p_0, p_1, p_5, p_{10} be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $P = \{p : p_{\mathbf{1}} = a \wedge c \leq p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leq d \vee p_{\mathbf{2}} = d \wedge a \leq p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leq b \vee p_{\mathbf{1}} = b \wedge c \leq p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leq d \vee p_{\mathbf{2}} = c \wedge a \leq p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leq b\}$ and $p_0 = [a,c]$ and $p_1 = [b,d]$ and $p_5 = [a,d]$ and $p_{10} = [b,c]$ and $f_1 = \langle p_0, p_5, p_1 \rangle$ and $f_2 = \langle p_0, p_{10}, p_1 \rangle$. Then f_1 is a special sequence and $\widetilde{\mathcal{L}}(f_1) = \mathcal{L}(p_0,p_5) \cup \mathcal{L}(p_5,p_1)$ and f_2 is a special sequence and $\widetilde{\mathcal{L}}(f_2) = \mathcal{L}(p_0,p_{10}) \cup \mathcal{L}(p_{10},p_1)$ and $P = \widetilde{\mathcal{L}}(f_1) \cup \widetilde{\mathcal{L}}(f_2)$ and $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) = \{p_0, p_1\}$ and $(f_1)_1 = p_0$ and $(f_1)_{\mathrm{len}\, f_1} = p_1$ and $(f_2)_1 = p_0$ and $(f_2)_{\mathrm{len}\, f_2} = p_1$.

(59) Let P, P_1, P_3 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, f_1, f_2 be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $P = \{p : p_{\mathbf{1}} = a \wedge c \leq p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leq d \vee p_{\mathbf{2}} = d \wedge a \leq p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leq b \vee p_{\mathbf{1}} = b \wedge c \leq p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leq d \vee p_{\mathbf{2}} = c \wedge a \leq p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leq b\}$ and $p_1 = [a,c]$ and $p_2 = [b,d]$ and $f_1 = \langle [a,c],[a,d],[b,d] \rangle$ and $f_2 = \langle [a,c],[b,c],[b,d] \rangle$ and $P_1 = \widetilde{\mathcal{L}}(f_1)$ and $P_3 = \widetilde{\mathcal{L}}(f_2)$. Then P_1 is an arc from p_1 to p_2 and P_3 is an arc from p_1 to p_2 and P_1 is non empty and P_3 is non empty and $P = P_1 \cup P_3$ and $P_1 \cap P_3 = \{p_1, p_2\}$.

(60) For all real numbers a, b, c, d such that $a < b$ and $c < d$ holds $\mathrm{Rectangle}(a,b,c,d)$ is a simple closed curve.

(61) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$, then $\mathrm{UpperArc}(K) = \mathcal{L}([a,c],[a,d]) \cup \mathcal{L}([a,d],[b,d])$.

(62) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$, then $\mathrm{LowerArc}(K) = \mathcal{L}([a,c],[b,c]) \cup \mathcal{L}([b,c],[b,d])$.

(63) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$. Then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction \mathrm{UpperArc}(K)$ such that

f is a homeomorphism and $f(0) = \mathrm{W}_{\mathrm{min}}(K)$ and $f(1) = \mathrm{E}_{\mathrm{max}}(K)$ and $\mathrm{rng}\, f = \mathrm{UpperArc}(K)$ and for every real number r such that $r \in [0, \frac{1}{2}]$ holds $f(r) = (1 - 2 \cdot r) \cdot [a,c] + 2 \cdot r \cdot [a,d]$ and for every real number r such that $r \in [\frac{1}{2}, 1]$ holds $f(r) = (1 - (2 \cdot r - 1)) \cdot [a,d] + (2 \cdot r - 1) \cdot [b,d]$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathcal{L}([a,c],[a,d])$ holds $0 \leq \frac{\frac{p_{\mathbf{2}} - c}{d - c}}{2}$ and $\frac{\frac{p_{\mathbf{2}} - c}{d - c}}{2} \leq 1$ and $f(\frac{\frac{p_{\mathbf{2}} - c}{d - c}}{2}) = p$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathcal{L}([a,d],[b,d])$ holds $0 \leq \frac{\frac{p_{\mathbf{1}} - a}{b - a}}{2} + \frac{1}{2}$ and $\frac{\frac{p_{\mathbf{1}} - a}{b - a}}{2} + \frac{1}{2} \leq 1$ and $f(\frac{\frac{p_{\mathbf{1}} - a}{b - a}}{2} + \frac{1}{2}) = p$.

(64) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$. Then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction\mathrm{LowerArc}(K)$ such that

f is a homeomorphism and $f(0) = \mathrm{E}_{\max}(K)$ and $f(1) = \mathrm{W}_{\min}(K)$ and $\mathrm{rng}\, f = \mathrm{LowerArc}(K)$ and for every real number r such that $r \in [0, \frac{1}{2}]$ holds $f(r) = (1-2\cdot r)\cdot[b,d]+2\cdot r\cdot[b,c]$ and for every real number r such that $r \in [\frac{1}{2}, 1]$ holds $f(r) = (1-(2\cdot r-1))\cdot[b,c]+(2\cdot r-1)\cdot[a,c]$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathcal{L}([b,d],[b,c])$ holds $0 \leq \frac{\frac{p_{\mathbf{2}}-d}{c-d}}{2}$ and $\frac{\frac{p_{\mathbf{2}}-d}{c-d}}{2} \leq 1$ and $f(\frac{\frac{p_{\mathbf{2}}-d}{c-d}}{2}) = p$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathcal{L}([b,c],[a,c])$ holds $0 \leq \frac{\frac{p_{\mathbf{1}}-b}{a-b}}{2}+\frac{1}{2}$ and $\frac{\frac{p_{\mathbf{1}}-b}{a-b}}{2}+\frac{1}{2} \leq 1$ and $f(\frac{\frac{p_{\mathbf{1}}-b}{a-b}}{2}+\frac{1}{2}) = p$.

(65) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([a,c],[a,d])$ and $p_2 \in \mathcal{L}([a,c],[a,d])$. Then $p_1 \leq_K p_2$ if and only if $(p_1)_{\mathbf{2}} \leq (p_2)_{\mathbf{2}}$.

(66) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([a,d],[b,d])$ and $p_2 \in \mathcal{L}([a,d],[b,d])$. Then $p_1 \leq_K p_2$ if and only if $(p_1)_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$.

(67) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([b,c],[b,d])$ and $p_2 \in \mathcal{L}([b,c],[b,d])$. Then $p_1 \leq_K p_2$ if and only if $(p_1)_{\mathbf{2}} \geq (p_2)_{\mathbf{2}}$.

(68) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([a,c],[b,c])$ and $p_2 \in \mathcal{L}([a,c],[b,c])$. Then $p_1 \leq_K p_2$ and $p_1 \neq \mathrm{W}_{\min}(K)$ if and only if $(p_1)_{\mathbf{1}} \geq (p_2)_{\mathbf{1}}$ and $p_2 \neq \mathrm{W}_{\min}(K)$.

(69) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([a,c],[a,d])$. Then $p_1 \leq_K p_2$ if and only if one of the following conditions is satisfied:

(i) $p_2 \in \mathcal{L}([a,c],[a,d])$ and $(p_1)_{\mathbf{2}} \leq (p_2)_{\mathbf{2}}$, or

(ii) $p_2 \in \mathcal{L}([a,d],[b,d])$, or

(iii) $p_2 \in \mathcal{L}([b,d],[b,c])$, or

(iv) $p_2 \in \mathcal{L}([b,c],[a,c])$ and $p_2 \neq \mathrm{W}_{\min}(K)$.

(70) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([a,d],[b,d])$. Then $p_1 \leq_K p_2$ if and only if one of the following conditions is satisfied:

(i) $p_2 \in \mathcal{L}([a,d],[b,d])$ and $(p_1)_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$, or

(ii) $p_2 \in \mathcal{L}([b,d],[b,c])$, or

(iii) $p_2 \in \mathcal{L}([b,c],[a,c])$ and $p_2 \neq \mathrm{W}_{\min}(K)$.

(71) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([b,d],[b,c])$. Then $p_1 \leq_K p_2$ if and only if one of the following conditions is satisfied:

(i) $p_2 \in \mathcal{L}([b,d],[b,c])$ and $(p_1)_{\mathbf{2}} \geq (p_2)_{\mathbf{2}}$, or

(ii) $p_2 \in \mathcal{L}([b,c],[a,c])$ and $p_2 \neq \mathrm{W}_{\min}(K)$.

(72) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([b,c],[a,c])$ and $p_1 \neq \mathrm{W}_{\min}(K)$. Then $p_1 \leq_K p_2$ if and only if the following conditions are satisfied:

(i) $p_2 \in \mathcal{L}([b,c],[a,c])$,

(ii) $(p_1)_{\mathbf{1}} \geq (p_2)_{\mathbf{1}}$, and

(iii) $p_2 \neq \mathrm{W}_{\min}(K)$.

(73) Let x be a set and a, b, c, d be real numbers. Suppose $x \in \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$. Then $x \in \mathcal{L}([a,c],[a,d])$ or $x \in \mathcal{L}([a,d],[b,d])$ or $x \in \mathcal{L}([b,d],[b,c])$ or $x \in \mathcal{L}([b,c],[a,c])$.

5. General Fashoda Theorem for Square

One can prove the following propositions:

(74) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $p_1 \leq_K p_2$ and $p_1 \in \mathcal{L}([-1,-1],[-1,1])$. Then $p_2 \in \mathcal{L}([-1,-1],[-1,1])$ and $(p_2)_{\mathbf{2}} \geq (p_1)_{\mathbf{2}}$ or $p_2 \in \mathcal{L}([-1,1],[1,1])$ or $p_2 \in \mathcal{L}([1,1],[1,-1])$ or $p_2 \in \mathcal{L}([1,-1],[-1,-1])$ and $p_2 \neq [-1,-1]$.

(75) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$ and $p_1 \in \mathcal{L}([-1,-1],[-1,1])$ and $(p_1)_{\mathbf{2}} \geq 0$ and $p_1 \leq_K p_2$. Then $f(p_1) \leq_P f(p_2)$.

(76) Let p_1, p_2, p_3 be points of $\mathcal{E}^2_{\mathrm{T}}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$ and $p_1 \in \mathcal{L}([-1,-1],[-1,1])$ and $(p_1)_{\mathbf{2}} \geq 0$ and $p_1 \leq_K p_2$ and $p_2 \leq_K p_3$. Then $f(p_2) \leq_P f(p_3)$.

(77) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. If $f = \mathrm{SqCirc}$ and $p_{\mathbf{1}} = -1$ and $p_{\mathbf{2}} < 0$, then $f(p)_{\mathbf{1}} < 0$ and $f(p)_{\mathbf{2}} < 0$.

(78) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. If $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$, then $f(p)_{\mathbf{1}} \geq 0$ iff $p_{\mathbf{1}} \geq 0$.

(79) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. If $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$, then $f(p)_{\mathbf{2}} \geq 0$ iff $p_{\mathbf{2}} \geq 0$.

(80) Let p, q be points of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. If $f = \mathrm{SqCirc}$ and $p \in \mathcal{L}([-1,-1],[-1,1])$ and $q \in \mathcal{L}([1,-1],[-1,-1])$, then $f(p)_{\mathbf{1}} \leq f(q)_{\mathbf{1}}$.

(81) Let p, q be points of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f = \mathrm{SqCirc}$ and $p \in \mathcal{L}([-1,-1],[-1,1])$ and $q \in \mathcal{L}([-1,-1],[-1,1])$ and $p_{\mathbf{2}} \geq q_{\mathbf{2}}$ and $p_{\mathbf{2}} < 0$. Then $f(p)_{\mathbf{2}} \geq f(q)_{\mathbf{2}}$.

(82) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$. Suppose $p_1 \leq_K p_2$ and $p_2 \leq_K p_3$ and $p_3 \leq_K p_4$. Then $f(p_1)$, $f(p_2)$, $f(p_3)$, $f(p_4)$ are in this order on P.

(83) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. If P is a simple closed curve and $p_1 \in P$ and $p_2 \in P$ and $p_1 \not\leq_P p_2$, then $p_2 \leq_P p_1$.

(84) Let p_1, p_2, p_3 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is a simple closed curve and $p_1 \in P$ and $p_2 \in P$ and $p_3 \in P$. Then $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ or $p_1 \leq_P p_3$ and $p_3 \leq_P p_2$ or $p_2 \leq_P p_1$ and $p_1 \leq_P p_3$ or $p_2 \leq_P p_3$ and $p_3 \leq_P p_1$ or $p_3 \leq_P p_1$ and $p_1 \leq_P p_2$ or $p_3 \leq_P p_2$ and $p_2 \leq_P p_1$.

(85) Let p_1, p_2, p_3 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is a simple closed curve and $p_1 \in P$ and $p_2 \in P$ and $p_3 \in P$ and $p_2 \leq_P p_3$. Then $p_1 \leq_P p_2$ or $p_2 \leq_P p_1$ and $p_1 \leq_P p_3$ or $p_3 \leq_P p_1$.

(86) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is a simple closed curve and $p_1 \in P$ and $p_2 \in P$ and $p_3 \in P$ and $p_4 \in P$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$. Then $p_1 \leq_P p_2$ or $p_2 \leq_P p_1$ and $p_1 \leq_P p_3$ or $p_3 \leq_P p_1$ and $p_1 \leq_P p_4$ or $p_4 \leq_P p_1$.

(87) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$ and $f(p_1) \leq_P f(p_2)$ and $f(p_2) \leq_P f(p_3)$ and $f(p_3) \leq_P f(p_4)$. Then p_1, p_2, p_3, p_4 are in this order on K.

(88) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$. Then p_1, p_2, p_3, p_4 are in this order on K if and only if $f(p_1)$, $f(p_2)$, $f(p_3)$, $f(p_4)$ are in this order on P.

(89) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, K be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and K_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(-1,1,-1,1)$ and p_1, p_2, p_3, p_4 are in this order on K. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $K_0 = \mathrm{ClosedInsideOfRectangle}(-1,1,-1,1)$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and $\mathrm{rng}\, f \subseteq K_0$ and $\mathrm{rng}\, g \subseteq K_0$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

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