JOURNAL OF FORMALIZED MATHEMATICS
Volume 14, Released 2002, Published 2003
Inst. of Computer Science, Univ. of Białystok

General Fashoda Meet Theorem for Unit Circle

Yatsuka Nakamura
Shinshu University
Nagano

Summary. Outside and inside Fashoda theorems are proven for points in general position on unit circle. Four points must be ordered in a sense of ordering for simple closed curve. For preparation of proof, the relation between the order and condition of coordinates of points on unit circle is discussed.

MML Identifier: JGRAPH_5.

WWW: http://mizar.org/JFM/Vol14/jgraph_5.html

The articles [1], [18], [11], [9], [17], [20], [8], [4], [5], [10], [2], [7], [12], [19], [16], [6], [3], [15], [14], and [13] provide the notation and terminology for this paper.

1. Preliminaries

In this paper x, a denote real numbers.

One can prove the following propositions:

(1) If $a \geq 0$ and $(x-a)\cdot(x+a) \geq 0$, then $-a \geq x$ or $x \geq a$.

(2) If $a \leq 0$ and $x < a$, then $x^2 > a^2$.

(3) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $|p| \leq 1$ holds $-1 \leq p_1$ and $p_1 \leq 1$ and $-1 \leq p_2$ and $p_2 \leq 1$.

(4) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $|p| \leq 1$ and $p_1 \neq 0$ and $p_2 \neq 0$ holds $-1 < p_1$ and $p_1 < 1$ and $-1 < p_2$ and $p_2 < 1$.

(5) Let a, b, d, e, r_3 be real numbers, P_1, P_2 be non empty metric structures, x be an element of P_1, and x_2 be an element of P_2. Suppose $d \leq a$ and $a \leq b$ and $b \leq e$ and $P_1 = [a, b]_{\mathrm{M}}$ and $P_2 = [d, e]_{\mathrm{M}}$ and $x = x_2$ and $x \in$ the carrier of P_1 and $x_2 \in$ the carrier of P_2. Then $\mathrm{Ball}(x, r_3) \subseteq \mathrm{Ball}(x_2, r_3)$.

(6) Let a, b, d, e be real numbers and B be a subset of $[d, e]_{\mathrm{T}}$. If $d \leq a$ and $a \leq b$ and $b \leq e$ and $B = [a, b]$, then $[a, b]_{\mathrm{T}} = [d, e]_{\mathrm{T}} \restriction B$.

(7) For all real numbers a, b and for every subset B of $\mathbb{I}$ such that $0 \leq a$ and $a \leq b$ and $b \leq 1$ and $B = [a, b]$ holds $[a, b]_{\mathrm{T}} = \mathbb{I} \restriction B$.

(8) Let X be a topological structure, Y, Z be non empty topological structures, f be a map from X into Y, and h be a map from Y into Z. If h is a homeomorphism and f is continuous, then $h \cdot f$ is continuous.

(9) Let X, Y, Z be topological structures, f be a map from X into Y, and h be a map from Y into Z. If h is a homeomorphism and f is one-to-one, then $h \cdot f$ is one-to-one.

(10) Let X be a topological structure, S, V be non empty topological structures, B be a non empty subset of S, f be a map from X into $S{\restriction}B$, g be a map from S into V, and h be a map from X into V. If $h = g \cdot f$ and f is continuous and g is continuous, then h is continuous.

(11) Let a, b, d, e, s_1, s_2, t_1, t_2 be real numbers and h be a map from $[a, b]_{\rm T}$ into $[d, e]_{\rm T}$. Suppose h is a homeomorphism and $h(s_1) = t_1$ and $h(s_2) = t_2$ and $h(a) = d$ and $h(b) = e$ and $d \leq e$ and $t_1 \leq t_2$ and $s_1 \in [a,b]$ and $s_2 \in [a,b]$. Then $s_1 \leq s_2$.

(12) Let a, b, d, e, s_1, s_2, t_1, t_2 be real numbers and h be a map from $[a, b]_{\rm T}$ into $[d, e]_{\rm T}$. Suppose h is a homeomorphism and $h(s_1) = t_1$ and $h(s_2) = t_2$ and $h(a) = e$ and $h(b) = d$ and $e \geq d$ and $t_1 \geq t_2$ and $s_1 \in [a,b]$ and $s_2 \in [a,b]$. Then $s_1 \leq s_2$.

(13) For every natural number n holds $-0_{\mathcal{E}^n_{\rm T}} = 0_{\mathcal{E}^n_{\rm T}}$.

2. Fashoda Meet Theorems for Circle in Special Case

One can prove the following propositions:

(14) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $a \neq b$ and $c \neq d$ and $f(O)_{\mathbf{1}} = a$ and $c \leq f(O)_{\mathbf{2}}$ and $f(O)_{\mathbf{2}} \leq d$ and $f(I)_{\mathbf{1}} = b$ and $c \leq f(I)_{\mathbf{2}}$ and $f(I)_{\mathbf{2}} \leq d$ and $g(O)_{\mathbf{2}} = c$ and $a \leq g(O)_{\mathbf{1}}$ and $g(O)_{\mathbf{1}} \leq b$ and $g(I)_{\mathbf{2}} = d$ and $a \leq g(I)_{\mathbf{1}}$ and $g(I)_{\mathbf{1}} \leq b$ and for every point r of $\mathbb{I}$ holds $a \geq f(r)_{\mathbf{1}}$ or $f(r)_{\mathbf{1}} \geq b$ or $c \geq f(r)_{\mathbf{2}}$ or $f(r)_{\mathbf{2}} \geq d$ but $a \geq g(r)_{\mathbf{1}}$ or $g(r)_{\mathbf{1}} \geq b$ or $c \geq g(r)_{\mathbf{2}}$ or $g(r)_{\mathbf{2}} \geq d$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(15) Let f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose f is continuous and one-to-one. Then there exists a map f_2 from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$ such that $f_2(0) = f(1)$ and $f_2(1) = f(0)$ and $\operatorname{rng} f_2 = \operatorname{rng} f$ and f_2 is continuous and one-to-one.

In the sequel p, q are points of $\mathcal{E}^2_{\rm T}$.
The following propositions are true:

(16) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, C_0, K_1, K_2, K_3, K_4 be subsets of $\mathcal{E}^2_{\rm T}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leq 1\}$ and $K_1 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_1| = 1 \wedge (q_1)_{\mathbf{2}} \leq (q_1)_{\mathbf{1}} \wedge (q_1)_{\mathbf{2}} \geq -(q_1)_{\mathbf{1}}\}$ and $K_2 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_2| = 1 \wedge (q_2)_{\mathbf{2}} \geq (q_2)_{\mathbf{1}} \wedge (q_2)_{\mathbf{2}} \leq -(q_2)_{\mathbf{1}}\}$ and $K_3 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_3| = 1 \wedge (q_3)_{\mathbf{2}} \geq (q_3)_{\mathbf{1}} \wedge (q_3)_{\mathbf{2}} \geq -(q_3)_{\mathbf{1}}\}$ and $K_4 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_4| = 1 \wedge (q_4)_{\mathbf{2}} \leq (q_4)_{\mathbf{1}} \wedge (q_4)_{\mathbf{2}} \leq -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_2$ and $f(I) \in K_1$ and $g(O) \in K_3$ and $g(I) \in K_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(17) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, C_0, K_1, K_2, K_3, K_4 be subsets of $\mathcal{E}^2_{\rm T}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \geq 1\}$ and $K_1 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_1| = 1 \wedge (q_1)_{\mathbf{2}} \leq (q_1)_{\mathbf{1}} \wedge (q_1)_{\mathbf{2}} \geq -(q_1)_{\mathbf{1}}\}$ and $K_2 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_2| = 1 \wedge (q_2)_{\mathbf{2}} \geq (q_2)_{\mathbf{1}} \wedge (q_2)_{\mathbf{2}} \leq -(q_2)_{\mathbf{1}}\}$ and $K_3 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_3| = 1 \wedge (q_3)_{\mathbf{2}} \geq (q_3)_{\mathbf{1}} \wedge (q_3)_{\mathbf{2}} \geq -(q_3)_{\mathbf{1}}\}$ and $K_4 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_4| = 1 \wedge (q_4)_{\mathbf{2}} \leq (q_4)_{\mathbf{1}} \wedge (q_4)_{\mathbf{2}} \leq -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_2$ and $f(I) \in K_1$ and $g(O) \in K_4$ and $g(I) \in K_3$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(18) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, C_0, K_1, K_2, K_3, K_4 be subsets of $\mathcal{E}^2_{\rm T}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \geq 1\}$ and $K_1 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_1| = 1 \wedge (q_1)_{\mathbf{2}} \leq (q_1)_{\mathbf{1}} \wedge (q_1)_{\mathbf{2}} \geq -(q_1)_{\mathbf{1}}\}$ and $K_2 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_2| = 1 \wedge (q_2)_{\mathbf{2}} \geq (q_2)_{\mathbf{1}} \wedge (q_2)_{\mathbf{2}} \leq -(q_2)_{\mathbf{1}}\}$ and $K_3 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_3| = 1 \wedge (q_3)_{\mathbf{2}} \geq (q_3)_{\mathbf{1}} \wedge (q_3)_{\mathbf{2}} \geq -(q_3)_{\mathbf{1}}\}$ and $K_4 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q_4| =$

$1 \wedge (q_4)_{\mathbf{2}} \leq (q_4)_{\mathbf{1}} \wedge (q_4)_{\mathbf{2}} \leq -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_2$ and $f(I) \in K_1$ and $g(O) \in K_3$ and $g(I) \in K_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(19) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$ and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $C_0 = \{q : |q| \geq 1\}$ and f is continuous and one-to-one and g is continuous and one-to-one and $f(0) = [-1,0]$ and $f(1) = [1,0]$ and $g(1) = [0,1]$ and $g(0) = [0,-1]$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(20) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) $C_0 = \{p : |p| \geq 1\}$,

(ii) $|p_1| = 1$,

(iii) $|p_2| = 1$,

(iv) $|p_3| = 1$,

(v) $|p_4| = 1$, and

(vi) there exists a map h from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that h is a homeomorphism and $h^\circ C_0 \subseteq C_0$ and $h(p_1) = [-1,0]$ and $h(p_2) = [0,1]$ and $h(p_3) = [1,0]$ and $h(p_4) = [0,-1]$.

Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_4$ and $g(1) = p_2$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

3. Properties of Fan Morphisms

One can prove the following propositions:

(21) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} > 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphN(q), then $p_{\mathbf{2}} > 0$.

(22) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} \geq 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphN(q), then $p_{\mathbf{2}} \geq 0$.

(23) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} \geq 0$ and $\frac{q_{\mathbf{1}}}{|q|} < c_1$ and $|q| \neq 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphN(q), then $p_{\mathbf{2}} \geq 0$ and $p_{\mathbf{1}} < 0$.

(24) Let c_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} \geq 0$ and $(q_2)_{\mathbf{2}} \geq 0$ and $|q_1| \neq 0$ and $|q_2| \neq 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = c_1$-FanMorphN(q_1) and $p_2 = c_1$-FanMorphN(q_2), then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

(25) Let s_3 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_3$ and $s_3 < 1$ and $q_{\mathbf{1}} > 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = s_3$-FanMorphE(q), then $p_{\mathbf{1}} > 0$.

(26) Let s_3 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_3$ and $s_3 < 1$ and $q_{\mathbf{1}} \geq 0$ and $\frac{q_{\mathbf{2}}}{|q|} < s_3$ and $|q| \neq 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = s_3$-FanMorphE(q), then $p_{\mathbf{1}} \geq 0$ and $p_{\mathbf{2}} < 0$.

(27) Let s_3 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_3$ and $s_3 < 1$ and $(q_1)_{\mathbf{1}} \geq 0$ and $(q_2)_{\mathbf{1}} \geq 0$ and $|q_1| \neq 0$ and $|q_2| \neq 0$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < \frac{(q_2)_{\mathbf{2}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = s_3$-FanMorphE(q_1) and $p_2 = s_3$-FanMorphE(q_2), then $\frac{(p_1)_{\mathbf{2}}}{|p_1|} < \frac{(p_2)_{\mathbf{2}}}{|p_2|}$.

(28) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} < 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphS(q), then $p_{\mathbf{2}} < 0$.

(29) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} < 0$ and $\frac{q_{\mathbf{1}}}{|q|} > c_1$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphS(q), then $p_{\mathbf{2}} < 0$ and $p_{\mathbf{1}} > 0$.

(30) Let c_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} \leq 0$ and $(q_2)_{\mathbf{2}} \leq 0$ and $|q_1| \neq 0$ and $|q_2| \neq 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If $p_1 = c_1$-FanMorphS(q_1) and $p_2 = c_1$-FanMorphS(q_2), then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

4. Order of Points on Circle

We now state a number of propositions:

(31) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{q : |q| = 1\}$ holds W-bound$(P) = -1$ and E-bound$(P) = 1$ and S-bound$(P) = -1$ and N-bound$(P) = 1$.

(32) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{q : |q| = 1\}$ holds $\mathrm{W_{min}}(P) = [-1, 0]$.

(33) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{q : |q| = 1\}$ holds $\mathrm{E_{max}}(P) = [1, 0]$.

(34) For every map f from $\mathcal{E}^2_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of $\mathcal{E}^2_{\rm T}$ holds $f(p) = \mathrm{proj1}(p)$ holds f is continuous.

(35) For every map f from $\mathcal{E}^2_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of $\mathcal{E}^2_{\rm T}$ holds $f(p) = \mathrm{proj2}(p)$ holds f is continuous.

(36) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q| = 1\}$ holds UpperArc$(P) \subseteq P$ and LowerArc$(P) \subseteq P$.

(37) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q| = 1\}$. Then UpperArc$(P) = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p \in P \wedge p_{\mathbf{2}} \geq 0\}$.

(38) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|q| = 1\}$. Then LowerArc$(P) = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p \in P \wedge p_{\mathbf{2}} \leq 0\}$.

(39) Let a, b, d, e be real numbers. Suppose $a \leq b$ and $e > 0$. Then there exists a map f from $[a, b]_{\rm T}$ into $[e \cdot a + d, e \cdot b + d]_{\rm T}$ such that f is a homeomorphism and for every real number r such that $r \in [a, b]$ holds $f(r) = e \cdot r + d$.

(40) Let a, b, d, e be real numbers. Suppose $a \leq b$ and $e < 0$. Then there exists a map f from $[a, b]_{\rm T}$ into $[e \cdot b + d, e \cdot a + d]_{\rm T}$ such that f is a homeomorphism and for every real number r such that $r \in [a, b]$ holds $f(r) = e \cdot r + d$.

(41) There exists a map f from $\mathbb{I}$ into $[-1, 1]_{\rm T}$ such that f is a homeomorphism and for every real number r such that $r \in [0, 1]$ holds $f(r) = (-2) \cdot r + 1$ and $f(0) = 1$ and $f(1) = -1$.

(42) There exists a map f from $\mathbb{I}$ into $[-1, 1]_{\rm T}$ such that f is a homeomorphism and for every real number r such that $r \in [0, 1]$ holds $f(r) = 2 \cdot r - 1$ and $f(0) = -1$ and $f(1) = 1$.

(43) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$. Then there exists a map f from $[-1, 1]_{\rm T}$ into $(\mathcal{E}^2_{\rm T}) \restriction$LowerArc$(P)$ such that f is a homeomorphism and for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ LowerArc(P) holds $f(q_{\mathbf{1}}) = q$ and $f(-1) = \mathrm{W_{min}}(P)$ and $f(1) = \mathrm{E_{max}}(P)$.

(44) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$. Then there exists a map f from $[-1, 1]_{\rm T}$ into $(\mathcal{E}^2_{\rm T}) \restriction$UpperArc$(P)$ such that f is a homeomorphism and for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ UpperArc(P) holds $f(q_{\mathbf{1}}) = q$ and $f(-1) = \mathrm{W_{min}}(P)$ and $f(1) = \mathrm{E_{max}}(P)$.

(45) Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p| = 1\}$. Then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T}) \restriction$LowerArc$(P)$ such that

(i) f is a homeomorphism,

(ii) for all points q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ and for all real numbers r_1, r_2 such that $f(r_1) = q_1$ and $f(r_2) = q_2$ and $r_1 \in [0,1]$ and $r_2 \in [0,1]$ holds $r_1 < r_2$ iff $(q_1)_{\mathbf{1}} > (q_2)_{\mathbf{1}}$,

(iii) $f(0) = \mathrm{E}_{\max}(P)$, and

(iv) $f(1) = \mathrm{W}_{\min}(P)$.

(46) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$. Then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction \mathrm{UpperArc}(P)$ such that

(i) f is a homeomorphism,

(ii) for all points q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$ and for all real numbers r_1, r_2 such that $f(r_1) = q_1$ and $f(r_2) = q_2$ and $r_1 \in [0,1]$ and $r_2 \in [0,1]$ holds $r_1 < r_2$ iff $(q_1)_{\mathbf{1}} < (q_2)_{\mathbf{1}}$,

(iii) $f(0) = \mathrm{W}_{\min}(P)$, and

(iv) $f(1) = \mathrm{E}_{\max}(P)$.

(47) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. If $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_2 \in \mathrm{UpperArc}(P)$ and $p_1 \leq_P p_2$, then $p_1 \in \mathrm{UpperArc}(P)$.

(48) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_1 \neq p_2$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} < 0$ and $(p_2)_{\mathbf{2}} < 0$. Then $(p_1)_{\mathbf{1}} > (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$.

(49) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_1 \neq p_2$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} \geq 0$ and $(p_2)_{\mathbf{2}} \geq 0$. Then $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$.

(50) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_1 \neq p_2$ and $(p_1)_{\mathbf{2}} \geq 0$ and $(p_2)_{\mathbf{2}} \geq 0$. Then $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$.

(51) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_1 \neq p_2$ and $(p_1)_{\mathbf{2}} \leq 0$ and $(p_2)_{\mathbf{2}} \leq 0$ and $p_1 \neq \mathrm{W}_{\min}(P)$. Then $(p_1)_{\mathbf{1}} > (p_2)_{\mathbf{1}}$.

(52) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ but $(p_2)_{\mathbf{2}} \geq 0$ or $(p_2)_{\mathbf{1}} \geq 0$ but $p_1 \leq_P p_2$. Then $(p_1)_{\mathbf{2}} \geq 0$ or $(p_1)_{\mathbf{1}} \geq 0$.

(53) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_1 \neq p_2$ and $(p_1)_{\mathbf{1}} \geq 0$ and $(p_2)_{\mathbf{1}} \geq 0$. Then $(p_1)_{\mathbf{2}} > (p_2)_{\mathbf{2}}$.

(54) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} < 0$ and $(p_2)_{\mathbf{2}} < 0$ and $(p_1)_{\mathbf{1}} \geq (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} \leq (p_2)_{\mathbf{2}}$. Then $p_1 \leq_P p_2$.

(55) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{1}} > 0$ and $(p_2)_{\mathbf{1}} > 0$ and $(p_1)_{\mathbf{2}} < 0$ and $(p_2)_{\mathbf{2}} < 0$ and $(p_1)_{\mathbf{1}} \geq (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} \geq (p_2)_{\mathbf{2}}$. Then $p_1 \leq_P p_2$.

(56) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} \geq 0$ and $(p_2)_{\mathbf{2}} \geq 0$ and $(p_1)_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} \leq (p_2)_{\mathbf{2}}$. Then $p_1 \leq_P p_2$.

(57) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{2}} \geq 0$ and $(p_2)_{\mathbf{2}} \geq 0$ and $(p_1)_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$. Then $p_1 \leq_P p_2$.

(58) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{1}} \geq 0$ and $(p_2)_{\mathbf{1}} \geq 0$ and $(p_1)_{\mathbf{2}} \geq (p_2)_{\mathbf{2}}$. Then $p_1 \leq_P p_2$.

(59) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \in P$ and $p_2 \in P$ and $(p_1)_{\mathbf{2}} \leq 0$ and $(p_2)_{\mathbf{2}} \leq 0$ and $p_2 \neq \mathrm{W}_{\mathrm{min}}(P)$ and $(p_1)_{\mathbf{1}} \geq (p_2)_{\mathbf{1}}$. Then $p_1 \leq_P p_2$.

(60) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} \leq 0$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphS(q), then $p_{\mathbf{2}} \leq 0$.

(61) Let c_1 be a real number, p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$, and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $q_1 = c_1$-FanMorphS(p_1) and $q_2 = c_1$-FanMorphS(p_2). Then $q_1 \leq_P q_2$.

(62) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} \geq 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{2}} \geq 0$ and $(p_3)_{\mathbf{1}} < 0$ and $(p_3)_{\mathbf{2}} \geq 0$ and $(p_4)_{\mathbf{1}} < 0$ and $(p_4)_{\mathbf{2}} \geq 0$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_1)_{\mathbf{2}} < 0$ and $(q_2)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{2}} < 0$ and $(q_3)_{\mathbf{1}} < 0$ and $(q_3)_{\mathbf{2}} < 0$ and $(q_4)_{\mathbf{1}} < 0$ and $(q_4)_{\mathbf{2}} < 0$ and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_3 \leq_P q_4$.

(63) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$ and $(p_1)_{\mathbf{2}} \geq 0$ and $(p_2)_{\mathbf{2}} \geq 0$ and $(p_3)_{\mathbf{2}} \geq 0$ and $(p_4)_{\mathbf{2}} > 0$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_1)_{\mathbf{2}} \geq 0$ and $(q_2)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{2}} \geq 0$ and $(q_3)_{\mathbf{1}} < 0$ and $(q_3)_{\mathbf{2}} \geq 0$ and $(q_4)_{\mathbf{1}} < 0$ and $(q_4)_{\mathbf{2}} \geq 0$ and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_3 \leq_P q_4$.

(64) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$ and $(p_1)_{\mathbf{2}} \geq 0$ and $(p_2)_{\mathbf{2}} \geq 0$ and $(p_3)_{\mathbf{2}} \geq 0$ and $(p_4)_{\mathbf{2}} > 0$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_1)_{\mathbf{2}} < 0$ and $(q_2)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{2}} < 0$ and $(q_3)_{\mathbf{1}} < 0$ and $(q_3)_{\mathbf{2}} < 0$ and $(q_4)_{\mathbf{1}} < 0$ and $(q_4)_{\mathbf{2}} < 0$ and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_3 \leq_P q_4$.

(65) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$ and $(p_1)_{\mathbf{2}} \geq 0$ or $(p_1)_{\mathbf{1}} \geq 0$ and $(p_2)_{\mathbf{2}} \geq 0$ or $(p_2)_{\mathbf{1}} \geq 0$ and $(p_3)_{\mathbf{2}} \geq 0$ or $(p_3)_{\mathbf{1}} \geq 0$ and $(p_4)_{\mathbf{2}} > 0$ or $(p_4)_{\mathbf{1}} > 0$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{2}} \geq 0$ and $(q_2)_{\mathbf{2}} \geq 0$ and $(q_3)_{\mathbf{2}} \geq 0$ and $(q_4)_{\mathbf{2}} > 0$ and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_3 \leq_P q_4$.

(66) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$ and $(p_1)_{\mathbf{2}} \geq 0$ or $(p_1)_{\mathbf{1}} \geq 0$ and $(p_2)_{\mathbf{2}} \geq 0$ or $(p_2)_{\mathbf{1}} \geq 0$ and $(p_3)_{\mathbf{2}} \geq 0$ or $(p_3)_{\mathbf{1}} \geq 0$ and $(p_4)_{\mathbf{2}} > 0$ or $(p_4)_{\mathbf{1}} > 0$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_1)_{\mathbf{2}} < 0$ and $(q_2)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{2}} < 0$ and $(q_3)_{\mathbf{1}} < 0$ and $(q_3)_{\mathbf{2}} < 0$ and $(q_4)_{\mathbf{1}} < 0$ and $(q_4)_{\mathbf{2}} < 0$ and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_3 \leq_P q_4$.

(67) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_4 = \mathrm{W}_{\min}(P)$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_1)_{\mathbf{2}} < 0$ and $(q_2)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{2}} < 0$ and $(q_3)_{\mathbf{1}} < 0$ and $(q_3)_{\mathbf{2}} < 0$ and $(q_4)_{\mathbf{1}} < 0$ and $(q_4)_{\mathbf{2}} < 0$ and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_3 \leq_P q_4$.

(68) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and there exist points q_1, q_2, q_3, q_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that

f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $q_1 = f(p_1)$ and $q_2 = f(p_2)$ and $q_3 = f(p_3)$ and $q_4 = f(p_4)$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_1)_{\mathbf{2}} < 0$ and $(q_2)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{2}} < 0$ and $(q_3)_{\mathbf{1}} < 0$ and $(q_3)_{\mathbf{2}} < 0$ and $(q_4)_{\mathbf{1}} < 0$ and $(q_4)_{\mathbf{2}} < 0$ and $q_1 \leq_P q_2$ and $q_2 \leq_P q_3$ and $q_3 \leq_P q_4$.

5. General Fashoda Theorems

We now state several propositions:

(69) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$ and $p_1 \neq p_2$ and $p_2 \neq p_3$ and $p_3 \neq p_4$ and $(p_1)_{\mathbf{1}} < 0$ and $(p_2)_{\mathbf{1}} < 0$ and $(p_3)_{\mathbf{1}} < 0$ and $(p_4)_{\mathbf{1}} < 0$ and $(p_1)_{\mathbf{2}} < 0$ and $(p_2)_{\mathbf{2}} < 0$ and $(p_3)_{\mathbf{2}} < 0$ and $(p_4)_{\mathbf{2}} < 0$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $[-1,0] = f(p_1)$ and $[0,1] = f(p_2)$ and $[1,0] = f(p_3)$ and $[0,-1] = f(p_4)$.

(70) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$ and $p_1 \neq p_2$ and $p_2 \neq p_3$ and $p_3 \neq p_4$. Then there exists a map f from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ such that f is a homeomorphism and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|f(q)| = |q|$ and $[-1,0] = f(p_1)$ and $[0, 1] = f(p_2)$ and $[1,0] = f(p_3)$ and $[0,-1] = f(p_4)$.

(71) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leq 1\}$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and $\mathrm{rng}\, f \subseteq C_0$ and $\mathrm{rng}\, g \subseteq C_0$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(72) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leq 1\}$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_4$ and $g(1) = p_2$ and $\mathrm{rng}\, f \subseteq C_0$ and $\mathrm{rng}\, g \subseteq C_0$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(73) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and $p_2 \leq_P p_3$ and $p_3 \leq_P p_4$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \geq 1\}$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_4$ and $g(1) = p_2$ and $\mathrm{rng}\, f \subseteq C_0$ and $\mathrm{rng}\, g \subseteq C_0$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(74) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $p_1 \leq_P p_2$ and

$p_2 \leq_P p_3$ and $p_3 \leq_P p_4$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \geq 1\}$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[2] Leszek Borys. Paracompact and metrizable spaces. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/pcomps_1.html`.

[3] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/binop_1.html`.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[6] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Journal of Formalized Mathematics*, 9, 1997. `http://mizar.org/JFM/Vol9/pscomp_1.html`.

[7] Agata Darmochwał. Compact spaces. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/compts_1.html`.

[8] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/tops_2.html`.

[9] Agata Darmochwał. The Euclidean space. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/euclid.html`.

[10] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces — fundamental concepts. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/topmetr.html`.

[11] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[12] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/metric_1.html`.

[13] Yatsuka Nakamura. Fan homeomorphisms in the plane. *Journal of Formalized Mathematics*, 14, 2002. `http://mizar.org/JFM/Vol14/jgraph_4.html`.

[14] Yatsuka Nakamura and Andrzej Trybulec. A decomposition of simple closed curves and the order of their points. *Journal of Formalized Mathematics*, 9, 1997. `http://mizar.org/JFM/Vol9/jordan6.html`.

[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/pre_topc.html`.

[16] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rcomp_1.html`.

[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[18] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[19] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/square_1.html`.

[20] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received June 24, 2002

Published January 2, 2004