JOURNAL OF FORMALIZED MATHEMATICS
Volume 13, Released 2001, Published 2003
Inst. of Computer Science, Univ. of Białystok

On Outside Fashoda Meet Theorem

Yatsuka Nakamura
Shinshu University
Nagano

Summary. We have proven the "Fashoda Meet Theorem" in [14]. Here we prove the outside version of it. It says that if Britain and France intended to set the courses for ships to the opposite side of Africa, they must also meet.

MML Identifier: `JGRAPH_2`.

WWW: `http://mizar.org/JFM/Vol13/jgraph_2.html`

The articles [18], [21], [1], [19], [12], [22], [4], [2], [7], [6], [16], [11], [3], [13], [5], [20], [9], [10], [15], [8], and [17] provide the notation and terminology for this paper.

One can prove the following propositions:

(2)[1] For every real number a such that $1 \leq a$ holds $a \leq a^{\mathbf{2}}$.

(3) For every real number a such that $-1 \geq a$ holds $-a \leq a^{\mathbf{2}}$.

(4) For every real number a such that $-1 > a$ holds $-a < a^{\mathbf{2}}$.

(5) For all real numbers a, b such that $b^{\mathbf{2}} \leq a^{\mathbf{2}}$ and $a \geq 0$ holds $-a \leq b$ and $b \leq a$.

(6) For all real numbers a, b such that $b^{\mathbf{2}} < a^{\mathbf{2}}$ and $a \geq 0$ holds $-a < b$ and $b < a$.

(7) For all real numbers a, b such that $-a \leq b$ and $b \leq a$ holds $b^{\mathbf{2}} \leq a^{\mathbf{2}}$.

(8) For all real numbers a, b such that $-a < b$ and $b < a$ holds $b^{\mathbf{2}} < a^{\mathbf{2}}$.

In the sequel T, T_1, T_2, S denote non empty topological spaces.

Next we state a number of propositions:

(9) Let f be a map from T_1 into S, g be a map from T_2 into S, and F_1, F_2 be subsets of T. Suppose that T_1 is a subspace of T and T_2 is a subspace of T and $F_1 = \Omega_{(T_1)}$ and $F_2 = \Omega_{(T_2)}$ and $\Omega_{(T_1)} \cup \Omega_{(T_2)} = \Omega_T$ and F_1 is closed and F_2 is closed and f is continuous and g is continuous and for every set p such that $p \in \Omega_{(T_1)} \cap \Omega_{(T_2)}$ holds $f(p) = g(p)$. Then there exists a map h from T into S such that $h = f{+\cdot}g$ and h is continuous.

(10) Let n be a natural number, q_2 be a point of $\mathcal{E}^n$, q be a point of $\mathcal{E}^n_{\mathrm{T}}$, and r be a real number. If $q = q_2$, then $\mathrm{Ball}(q_2, r) = \{q_3; q_3$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $|q - q_3| < r\}$.

(11) $(0_{\mathcal{E}^2_{\mathrm{T}}})_{\mathbf{1}} = 0$ and $(0_{\mathcal{E}^2_{\mathrm{T}}})_{\mathbf{2}} = 0$.

(12) $1.\mathrm{REAL}\,2 = \langle 1, 1 \rangle$.

(13) $(1.\mathrm{REAL}\,2)_{\mathbf{1}} = 1$ and $(1.\mathrm{REAL}\,2)_{\mathbf{2}} = 1$.

[1] The proposition (1) has been removed.

(14) dom proj1 = the carrier of $\mathcal{E}^2_\mathrm{T}$ and dom proj1 = $\mathcal{R}^2$.

(15) dom proj2 = the carrier of $\mathcal{E}^2_\mathrm{T}$ and dom proj2 = $\mathcal{R}^2$.

(16) proj1 is a map from $\mathcal{E}^2_\mathrm{T}$ into $\mathbb{R}^\mathbf{1}$.

(17) proj2 is a map from $\mathcal{E}^2_\mathrm{T}$ into $\mathbb{R}^\mathbf{1}$.

(18) For every point p of $\mathcal{E}^2_\mathrm{T}$ holds $p = [\mathrm{proj1}(p), \mathrm{proj2}(p)]$.

(19) For every subset B of $\mathcal{E}^2_\mathrm{T}$ such that $B = \{0_{\mathcal{E}^2_\mathrm{T}}\}$ holds $B^\mathrm{c} \neq \emptyset$ and (the carrier of $\mathcal{E}^2_\mathrm{T}$) $\setminus B \neq \emptyset$.

(20) Let X, Y be non empty topological spaces and f be a map from X into Y. Then f is continuous if and only if for every point p of X and for every subset V of Y such that $f(p) \in V$ and V is open there exists a subset W of X such that $p \in W$ and W is open and $f^\circ W \subseteq V$.

(21) Let p be a point of $\mathcal{E}^2_\mathrm{T}$ and G be a subset of $\mathcal{E}^2_\mathrm{T}$. Suppose G is open and $p \in G$. Then there exists a real number r such that $r > 0$ and $\{q; q$ ranges over points of $\mathcal{E}^2_\mathrm{T}$: $p_\mathbf{1} - r < q_\mathbf{1} \wedge q_\mathbf{1} < p_\mathbf{1} + r \wedge p_\mathbf{2} - r < q_\mathbf{2} \wedge q_\mathbf{2} < p_\mathbf{2} + r\} \subseteq G$.

(22) Let X, Y, Z be non empty topological spaces, B be a subset of Y, C be a subset of Z, f be a map from X into Y, and h be a map from $Y\restriction B$ into $Z\restriction C$. Suppose f is continuous and h is continuous and $\mathrm{rng}\, f \subseteq B$ and $B \neq \emptyset$ and $C \neq \emptyset$. Then there exists a map g from X into Z such that g is continuous and $g = h \cdot f$.

In the sequel p, q are points of $\mathcal{E}^2_\mathrm{T}$.

The function OutInSq from (the carrier of $\mathcal{E}^2_\mathrm{T}$) $\setminus \{0_{\mathcal{E}^2_\mathrm{T}}\}$ into (the carrier of $\mathcal{E}^2_\mathrm{T}$) $\setminus \{0_{\mathcal{E}^2_\mathrm{T}}\}$ is defined by the condition (Def. 1).

(Def. 1) Let p be a point of $\mathcal{E}^2_\mathrm{T}$ such that $p \neq 0_{\mathcal{E}^2_\mathrm{T}}$. Then

(i) if $p_\mathbf{2} \leq p_\mathbf{1}$ and $-p_\mathbf{1} \leq p_\mathbf{2}$ or $p_\mathbf{2} \geq p_\mathbf{1}$ and $p_\mathbf{2} \leq -p_\mathbf{1}$, then $\mathrm{OutInSq}(p) = [\frac{1}{p_\mathbf{1}}, \frac{\frac{p_\mathbf{2}}{p_\mathbf{1}}}{p_\mathbf{1}}]$, and

(ii) if $p_\mathbf{2} \not\leq p_\mathbf{1}$ or $-p_\mathbf{1} \not\leq p_\mathbf{2}$ and if $p_\mathbf{2} \not\geq p_\mathbf{1}$ or $p_\mathbf{2} \not\leq -p_\mathbf{1}$, then $\mathrm{OutInSq}(p) = [\frac{\frac{p_\mathbf{1}}{p_\mathbf{2}}}{p_\mathbf{2}}, \frac{1}{p_\mathbf{2}}]$.

The following propositions are true:

(23) Let p be a point of $\mathcal{E}^2_\mathrm{T}$. Suppose $p_\mathbf{2} \not\leq p_\mathbf{1}$ or $-p_\mathbf{1} \not\leq p_\mathbf{2}$ but $p_\mathbf{2} \not\geq p_\mathbf{1}$ or $p_\mathbf{2} \not\leq -p_\mathbf{1}$. Then $p_\mathbf{1} \leq p_\mathbf{2}$ and $-p_\mathbf{2} \leq p_\mathbf{1}$ or $p_\mathbf{1} \geq p_\mathbf{2}$ and $p_\mathbf{1} \leq -p_\mathbf{2}$.

(24) Let p be a point of $\mathcal{E}^2_\mathrm{T}$ such that $p \neq 0_{\mathcal{E}^2_\mathrm{T}}$. Then

(i) if $p_\mathbf{1} \leq p_\mathbf{2}$ and $-p_\mathbf{2} \leq p_\mathbf{1}$ or $p_\mathbf{1} \geq p_\mathbf{2}$ and $p_\mathbf{1} \leq -p_\mathbf{2}$, then $\mathrm{OutInSq}(p) = [\frac{\frac{p_\mathbf{1}}{p_\mathbf{2}}}{p_\mathbf{2}}, \frac{1}{p_\mathbf{2}}]$, and

(ii) if $p_\mathbf{1} \not\leq p_\mathbf{2}$ or $-p_\mathbf{2} \not\leq p_\mathbf{1}$ and if $p_\mathbf{1} \not\geq p_\mathbf{2}$ or $p_\mathbf{1} \not\leq -p_\mathbf{2}$, then $\mathrm{OutInSq}(p) = [\frac{1}{p_\mathbf{1}}, \frac{\frac{p_\mathbf{2}}{p_\mathbf{1}}}{p_\mathbf{1}}]$.

(25) Let D be a subset of $\mathcal{E}^2_\mathrm{T}$ and K_0 be a subset of $(\mathcal{E}^2_\mathrm{T})\restriction D$. Suppose $K_0 = \{p : (p_\mathbf{2} \leq p_\mathbf{1} \wedge -p_\mathbf{1} \leq p_\mathbf{2} \vee p_\mathbf{2} \geq p_\mathbf{1} \wedge p_\mathbf{2} \leq -p_\mathbf{1}) \wedge p \neq 0_{\mathcal{E}^2_\mathrm{T}}\}$. Then $\mathrm{rng}(\mathrm{OutInSq}\restriction K_0) \subseteq$ the carrier of $(\mathcal{E}^2_\mathrm{T})\restriction D\restriction K_0$.

(26) Let D be a subset of $\mathcal{E}^2_\mathrm{T}$ and K_0 be a subset of $(\mathcal{E}^2_\mathrm{T})\restriction D$. Suppose $K_0 = \{p : (p_\mathbf{1} \leq p_\mathbf{2} \wedge -p_\mathbf{2} \leq p_\mathbf{1} \vee p_\mathbf{1} \geq p_\mathbf{2} \wedge p_\mathbf{1} \leq -p_\mathbf{2}) \wedge p \neq 0_{\mathcal{E}^2_\mathrm{T}}\}$. Then $\mathrm{rng}(\mathrm{OutInSq}\restriction K_0) \subseteq$ the carrier of $(\mathcal{E}^2_\mathrm{T})\restriction D\restriction K_0$.

(27) Let K_1 be a set and D be a non empty subset of $\mathcal{E}^2_\mathrm{T}$. Suppose $K_1 = \{p; p$ ranges over points of $\mathcal{E}^2_\mathrm{T}$: $(p_\mathbf{2} \leq p_\mathbf{1} \wedge -p_\mathbf{1} \leq p_\mathbf{2} \vee p_\mathbf{2} \geq p_\mathbf{1} \wedge p_\mathbf{2} \leq -p_\mathbf{1}) \wedge p \neq 0_{\mathcal{E}^2_\mathrm{T}}\}$ and $D^\mathrm{c} = \{0_{\mathcal{E}^2_\mathrm{T}}\}$. Then K_1 is a non empty subset of $(\mathcal{E}^2_\mathrm{T})\restriction D$ and a non empty subset of $\mathcal{E}^2_\mathrm{T}$.

(28) Let K_1 be a set and D be a non empty subset of $\mathcal{E}^2_\mathrm{T}$. Suppose $K_1 = \{p; p$ ranges over points of $\mathcal{E}^2_\mathrm{T}$: $(p_\mathbf{1} \leq p_\mathbf{2} \wedge -p_\mathbf{2} \leq p_\mathbf{1} \vee p_\mathbf{1} \geq p_\mathbf{2} \wedge p_\mathbf{1} \leq -p_\mathbf{2}) \wedge p \neq 0_{\mathcal{E}^2_\mathrm{T}}\}$ and $D^\mathrm{c} = \{0_{\mathcal{E}^2_\mathrm{T}}\}$. Then K_1 is a non empty subset of $(\mathcal{E}^2_\mathrm{T})\restriction D$ and a non empty subset of $\mathcal{E}^2_\mathrm{T}$.

(29) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 + r_2$ and g is continuous.

(30) Let X be a non empty topological space and a be a real number. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X holds $g(p) = a$ and g is continuous.

(31) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 - r_2$ and g is continuous.

(32) Let X be a non empty topological space and f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = r_1 \cdot r_1$ and g is continuous.

(33) Let X be a non empty topological space, f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$, and a be a real number. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = a \cdot r_1$ and g is continuous.

(34) Let X be a non empty topological space, f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$, and a be a real number. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = r_1 + a$ and g is continuous.

(35) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_1 \cdot r_2$ and g is continuous.

(36) Let X be a non empty topological space and f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and for every point q of X holds $f_1(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = \frac{1}{r_1}$ and g is continuous.

(37) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = \frac{r_1}{r_2}$ and g is continuous.

(38) Let X be a non empty topological space and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = \frac{\frac{r_1}{r_2}}{r_2}$, and

(ii) g is continuous.

(39) Let K_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}){\restriction}K_0$ into $\mathbb{R}^{\mathbf{1}}$. If for every point p of $(\mathcal{E}^2_{\mathrm{T}}){\restriction}K_0$ holds $f(p) = \mathrm{proj1}(p)$, then f is continuous.

(40) Let K_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}){\restriction}K_0$ into $\mathbb{R}^{\mathbf{1}}$. If for every point p of $(\mathcal{E}^2_{\mathrm{T}}){\restriction}K_0$ holds $f(p) = \mathrm{proj2}(p)$, then f is continuous.

(41) Let K_2 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}}){\restriction}K_2$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}){\restriction}K_2$ holds $f(p) = \frac{1}{p_{\mathbf{1}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}){\restriction}K_2$ holds $q_{\mathbf{1}} \neq 0$.

Then f is continuous.

(42) Let K_2 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_2$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_2$ holds $f(p) = \frac{1}{p_{\mathbf{2}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_2$ holds $q_{\mathbf{2}} \neq 0$.

Then f is continuous.

(43) Let K_2 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_2$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_2$ holds $f(p) = \frac{\frac{p_{\mathbf{2}}}{p_{\mathbf{1}}}}{p_{\mathbf{1}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_2$ holds $q_{\mathbf{1}} \neq 0$.

Then f is continuous.

(44) Let K_2 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_2$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_2$ holds $f(p) = \frac{\frac{p_{\mathbf{1}}}{p_{\mathbf{2}}}}{p_{\mathbf{2}}}$, and

(ii) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction K_2$ holds $q_{\mathbf{2}} \neq 0$.

Then f is continuous.

(45) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$, and f_1, f_2 be maps from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_0$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) f_1 is continuous,

(ii) f_2 is continuous,

(iii) $K_0 \neq \emptyset$,

(iv) $B_0 \neq \emptyset$, and

(v) for all real numbers x, y, r, s such that $[x,y] \in K_0$ and $r = f_1([x,y])$ and $s = f_2([x,y])$ holds $f([x,y]) = [r,s]$.

Then f is continuous.

(46) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$. Suppose $f = \mathrm{OutInSq}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : (p_{\mathbf{2}} \leq p_{\mathbf{1}} \wedge -p_{\mathbf{1}} \leq p_{\mathbf{2}} \vee p_{\mathbf{2}} \geq p_{\mathbf{1}} \wedge p_{\mathbf{2}} \leq -p_{\mathbf{1}}) \wedge p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then f is continuous.

(47) Let K_0, B_0 be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$. Suppose $f = \mathrm{OutInSq}\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : (p_{\mathbf{1}} \leq p_{\mathbf{2}} \wedge -p_{\mathbf{2}} \leq p_{\mathbf{1}} \vee p_{\mathbf{1}} \geq p_{\mathbf{2}} \wedge p_{\mathbf{1}} \leq -p_{\mathbf{2}}) \wedge p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then f is continuous.

In this article we present several logical schemes. The scheme *TopSubset* concerns a unary predicate $\mathcal{P}$, and states that:

$\{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $\mathcal{P}[p]\}$ is a subset of $\mathcal{E}^2_{\mathrm{T}}$

for all values of the parameters.

The scheme *TopCompl* deals with a subset $\mathcal{A}$ of $\mathcal{E}^2_{\mathrm{T}}$ and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{A}^{\mathrm{c}} = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: not $\mathcal{P}[p]\}$

provided the parameters satisfy the following condition:

- $\mathcal{A} = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $\mathcal{P}[p]\}$.

The scheme *ClosedSubset* deals with two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding real numbers, and states that:

$\{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $\mathcal{F}(p) \leq \mathcal{G}(p)\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$

provided the parameter meets the following requirements:

- For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathcal{F}(p-q) = \mathcal{F}(p) - \mathcal{F}(q)$ and $\mathcal{G}(p-q) = \mathcal{G}(p) - \mathcal{G}(q)$, and
- For all points p, q of $\mathcal{E}^2_{\mathrm{T}}$ holds $|p-q|^{\mathbf{2}} = \mathcal{F}(p-q)^{\mathbf{2}} + \mathcal{G}(p-q)^{\mathbf{2}}$.

Next we state several propositions:

(48) Let B_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$, K_0 be a subset of $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$. Suppose $f = \mathrm{OutInSq}\restriction K_0$ and $B_0 = (\text{the carrier of } \mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : (p_{\mathbf{2}} \leq p_{\mathbf{1}} \wedge -p_{\mathbf{1}} \leq p_{\mathbf{2}} \vee p_{\mathbf{2}} \geq p_{\mathbf{1}} \wedge p_{\mathbf{2}} \leq -p_{\mathbf{1}}) \wedge p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then f is continuous and K_0 is closed.

(49) Let B_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$, K_0 be a subset of $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B_0$. Suppose $f = \mathrm{OutInSq}\restriction K_0$ and $B_0 = (\text{the carrier of } \mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : (p_{\mathbf{1}} \leq p_{\mathbf{2}} \wedge -p_{\mathbf{2}} \leq p_{\mathbf{1}} \vee p_{\mathbf{1}} \geq p_{\mathbf{2}} \wedge p_{\mathbf{1}} \leq -p_{\mathbf{2}}) \wedge p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then f is continuous and K_0 is closed.

(50) Let D be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $D^{\mathrm{c}} = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then there exists a map h from $(\mathcal{E}^2_{\mathrm{T}})\restriction D$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction D$ such that $h = \mathrm{OutInSq}$ and h is continuous.

(51) Let B, K_0, K_3 be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) $B = \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$,

(ii) $K_0 = \{p : -1 < p_{\mathbf{1}} \wedge p_{\mathbf{1}} < 1 \wedge -1 < p_{\mathbf{2}} \wedge p_{\mathbf{2}} < 1\}$, and

(iii) $K_3 = \{q : -1 = q_{\mathbf{1}} \wedge -1 \leq q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leq 1 \vee q_{\mathbf{1}} = 1 \wedge -1 \leq q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leq 1 \vee -1 = q_{\mathbf{2}} \wedge -1 \leq q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leq 1 \vee 1 = q_{\mathbf{2}} \wedge -1 \leq q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leq 1\}$.

Then there exists a map f from $(\mathcal{E}^2_{\mathrm{T}})\restriction B^{\mathrm{c}}$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction B^{\mathrm{c}}$ such that

(iv) f is continuous and one-to-one,

(v) for every point t of $\mathcal{E}^2_{\mathrm{T}}$ such that $t \in K_0$ and $t \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$ holds $f(t) \notin K_0 \cup K_3$,

(vi) for every point r of $\mathcal{E}^2_{\mathrm{T}}$ such that $r \notin K_0 \cup K_3$ holds $f(r) \in K_0$, and

(vii) for every point s of $\mathcal{E}^2_{\mathrm{T}}$ such that $s \in K_3$ holds $f(s) = s$.

(52) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, K_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $K_0 = \{p : -1 < p_{\mathbf{1}} \wedge p_{\mathbf{1}} < 1 \wedge -1 < p_{\mathbf{2}} \wedge p_{\mathbf{2}} < 1\}$ and $f(O)_{\mathbf{1}} = -1$ and $f(I)_{\mathbf{1}} = 1$ and $-1 \leq f(O)_{\mathbf{2}}$ and $f(O)_{\mathbf{2}} \leq 1$ and $-1 \leq f(I)_{\mathbf{2}}$ and $f(I)_{\mathbf{2}} \leq 1$ and $g(O)_{\mathbf{2}} = -1$ and $g(I)_{\mathbf{2}} = 1$ and $-1 \leq g(O)_{\mathbf{1}}$ and $g(O)_{\mathbf{1}} \leq 1$ and $-1 \leq g(I)_{\mathbf{1}}$ and $g(I)_{\mathbf{1}} \leq 1$ and $\mathrm{rng} f$ misses K_0 and $\mathrm{rng}\, g$ misses K_0. Then $\mathrm{rng} f$ meets $\mathrm{rng}\, g$.

(53) Let A, B, C, D be real numbers and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that for every point t of $\mathcal{E}^2_{\mathrm{T}}$ holds $f(t) = [A \cdot t_{\mathbf{1}} + B, C \cdot t_{\mathbf{2}} + D]$. Then f is continuous.

Let A, B, C, D be real numbers. The functor $\mathrm{AffineMap}(A,B,C,D)$ yields a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 2) For every point t of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\mathrm{AffineMap}(A,B,C,D))(t) = [A \cdot t_{\mathbf{1}} + B, C \cdot t_{\mathbf{2}} + D]$.

Let a, b, c, d be real numbers. One can check that $\mathrm{AffineMap}(a,b,c,d)$ is continuous.

Next we state several propositions:

(54) For all real numbers A, B, C, D such that $A > 0$ and $C > 0$ holds $\mathrm{AffineMap}(A,B,C,D)$ is one-to-one.

(55) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $f(O)_{\mathbf{1}} = a$ and $f(I)_{\mathbf{1}} = b$ and $c \leq f(O)_{\mathbf{2}}$ and $f(O)_{\mathbf{2}} \leq d$ and $c \leq f(I)_{\mathbf{2}}$ and $f(I)_{\mathbf{2}} \leq d$ and $g(O)_{\mathbf{2}} = c$ and $g(I)_{\mathbf{2}} = d$ and $a \leq g(O)_{\mathbf{1}}$ and $g(O)_{\mathbf{1}} \leq b$ and $a \leq g(I)_{\mathbf{1}}$ and $g(I)_{\mathbf{1}} \leq b$ and $a < b$ and $c < d$ and it is not true that there exists a point r of $\mathbb{I}$ such that $a < f(r)_{\mathbf{1}}$ and $f(r)_{\mathbf{1}} < b$ and $c < f(r)_{\mathbf{2}}$ and $f(r)_{\mathbf{2}} < d$ and it is not true that there exists a point r of $\mathbb{I}$ such that $a < g(r)_{\mathbf{1}}$ and $g(r)_{\mathbf{1}} < b$ and $c < g(r)_{\mathbf{2}}$ and $g(r)_{\mathbf{2}} < d$. Then $\mathrm{rng} f$ meets $\mathrm{rng}\, g$.

(56)(i) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $(p_7)_{\mathbf{2}} \leq (p_7)_{\mathbf{1}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$, and

(ii) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $(p_7)_{\mathbf{1}} \leq (p_7)_{\mathbf{2}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

(57)(i) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $-(p_7)_{\mathbf{1}} \leq (p_7)_{\mathbf{2}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$, and

(ii) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $(p_7)_{\mathbf{2}} \leq -(p_7)_{\mathbf{1}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

(58)(i) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $-(p_7)_{\mathbf{2}} \leq (p_7)_{\mathbf{1}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$, and

(ii) $\{p_7; p_7$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $(p_7)_{\mathbf{1}} \leq -(p_7)_{\mathbf{2}}\}$ is a closed subset of $\mathcal{E}^2_{\mathrm{T}}$.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[3] Leszek Borys. Paracompact and metrizable spaces. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/pcomps_1.html`.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[6] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[7] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_2.html`.

[8] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/funct_4.html`.

[9] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Journal of Formalized Mathematics*, 9, 1997. `http://mizar.org/JFM/Vol9/pscomp_1.html`.

[10] Agata Darmochwał. The Euclidean space. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/euclid.html`.

[11] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces — fundamental concepts. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/topmetr.html`.

[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[13] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/metric_1.html`.

[14] Yatsuka Nakamura. Graph theoretical properties of arcs in the plane and Fashoda Meet Theorem. *Journal of Formalized Mathematics*, 10, 1998. `http://mizar.org/JFM/Vol10/jgraph_1.html`.

[15] Yatsuka Nakamura, Andrzej Trybulec, and Czesław Byliński. Bounded domains and unbounded domains. *Journal of Formalized Mathematics*, 11, 1999. `http://mizar.org/JFM/Vol11/jordan2c.html`.

[16] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/pre_topc.html`.

[17] Piotr Rudnicki and Andrzej Trybulec. On same equivalents of well-foundedness. *Journal of Formalized Mathematics*, 9, 1997. `http://mizar.org/JFM/Vol9/wellfnd1.html`.

[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[19] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[20] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/square_1.html`.

[21] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[22] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received July 16, 2001

Published January 2, 2004