JOURNAL OF FORMALIZED MATHEMATICS
Volume 10, Released 1998, Published 2003
Inst. of Computer Science, Univ. of Białystok

Graph Theoretical Properties of Arcs in the Plane and Fashoda Meet Theorem

Yatsuka Nakamura
Shinshu University
Nagano

Summary. We define a graph on an abstract set, edges of which are pairs of any two elements. For any finite sequence of a plane, we give a definition of nodic, which means that edges by a finite sequence are crossed only at terminals. If the first point and the last point of a finite sequence differs, simpleness as a chain and nodic condition imply unfoldedness and s.n.c. condition. We generalize Goboard Theorem, proved by us before, to a continuous case. We call this Fashoda Meet Theorem, which was taken from Fashoda incident of 100 years ago.

MML Identifier: `JGRAPH_1`.

WWW: `http://mizar.org/JFM/Vol10/jgraph_1.html`

The articles [32], [10], [36], [3], [33], [20], [37], [8], [9], [4], [11], [16], [1], [2], [21], [28], [25], [35], [26], [19], [27], [29], [24], [23], [18], [6], [14], [5], [15], [22], [30], [34], [13], [12], [31], [17], and [7] provide the notation and terminology for this paper.

1. A Graph by Cartesian Product

For simplicity, we follow the rules: G is a graph, v_1 is a finite sequence of elements of the vertices of G, I_1 is an oriented chain of G, n, m, k, i, j are natural numbers, and r, r_1, r_2 are real numbers.

One can prove the following propositions:

(2)[1] $\sqrt{{r_1}^2 + {r_2}^2} \leq |r_1| + |r_2|$.

(3) $|r_1| \leq \sqrt{{r_1}^2 + {r_2}^2}$ and $|r_2| \leq \sqrt{{r_1}^2 + {r_2}^2}$.

(4) Let given v_1. Suppose I_1 is Simple and v_1 is oriented vertex seq of I_1. Let given n, m. If $1 \leq n$ and $n < m$ and $m \leq \operatorname{len} v_1$ and $v_1(n) = v_1(m)$, then $n = 1$ and $m = \operatorname{len} v_1$.

Let X be a set. The functor PGraph$X$ yielding a multi graph structure is defined by:

(Def. 1) $\mathrm{PGraph}X = \langle X, [:X, X:], \pi_1(X \times X), \pi_2(X \times X)\rangle$.

One can prove the following propositions:

(5) For every non empty set X holds PGraph$X$ is a graph.

(6) For every set X holds the vertices of $\mathrm{PGraph}X = X$.

[1] The proposition (1) has been removed.

Let f be a finite sequence. The functor PairF f yielding a finite sequence is defined as follows:

(Def. 2) $\operatorname{len}\operatorname{PairF} f = \operatorname{len} f -' 1$ and for every natural number i such that $1 \leq i$ and $i < \operatorname{len} f$ holds $(\operatorname{PairF} f)(i) = \langle f(i), f(i+1) \rangle$.

In the sequel X is a non empty set.

Let X be a non empty set. Observe that PGraph X is graph-like.

Next we state two propositions:

(7) Every finite sequence of elements of X is a finite sequence of elements of the vertices of PGraph X.

(8) For every finite sequence f of elements of X holds PairF f is a finite sequence of elements of the edges of PGraph X.

Let X be a non empty set and let f be a finite sequence of elements of X. Then PairF f is a finite sequence of elements of the edges of PGraph X.

Next we state two propositions:

(9) Let n be a natural number and f be a finite sequence of elements of X. If $1 \leq n$ and $n \leq \operatorname{len}\operatorname{PairF} f$, then $(\operatorname{PairF} f)(n) \in$ the edges of PGraph X.

(10) For every finite sequence f of elements of X holds PairF f is an oriented chain of PGraph X.

Let X be a non empty set and let f be a finite sequence of elements of X. Then PairF f is an oriented chain of PGraph X.

One can prove the following proposition

(11) Let f be a finite sequence of elements of X and f_1 be a finite sequence of elements of the vertices of PGraph X. If $\operatorname{len} f \geq 1$ and $f = f_1$, then f_1 is oriented vertex seq of PairF f.

2. Shortcuts of Finite Sequences in Plane

Let X be a non empty set and let f, g be finite sequences of elements of X. We say that g is Shortcut of f if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $f(1) = g(1)$,

(ii) $f(\operatorname{len} f) = g(\operatorname{len} g)$, and

(iii) there exists a FinSubsequence f_2 of PairF f and there exists a FinSubsequence f_3 of f and there exists an oriented simple chain s_1 of PGraph X and there exists a finite sequence g_1 of elements of the vertices of PGraph X such that $\operatorname{Seq} f_2 = s_1$ and $\operatorname{Seq} f_3 = g$ and $g_1 = g$ and g_1 is oriented vertex seq of s_1.

Next we state four propositions:

(12) For all finite sequences f, g of elements of X such that g is Shortcut of f holds $1 \leq \operatorname{len} g$ and $\operatorname{len} g \leq \operatorname{len} f$.

(13) Let f be a finite sequence of elements of X. Suppose $\operatorname{len} f \geq 1$. Then there exists a finite sequence g of elements of X such that g is Shortcut of f.

(14) For all finite sequences f, g of elements of X such that g is Shortcut of f holds $\operatorname{rng}\operatorname{PairF} g \subseteq \operatorname{rng}\operatorname{PairF} f$.

(15) Let f, g be finite sequences of elements of X. Suppose $f(1) \neq f(\operatorname{len} f)$ and g is Shortcut of f. Then g is one-to-one and $\operatorname{rng}\operatorname{PairF} g \subseteq \operatorname{rng}\operatorname{PairF} f$ and $g(1) = f(1)$ and $g(\operatorname{len} g) = f(\operatorname{len} f)$.

Let us consider n and let I_1 be a finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$. We say that I_1 is nodic if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let given i, j. Suppose $\mathcal{L}(I_1,i)$ meets $\mathcal{L}(I_1,j)$. Then $\mathcal{L}(I_1,i)\cap\mathcal{L}(I_1,j)=\{I_1(i)\}$ but $I_1(i)=I_1(j)$ or $I_1(i)=I_1(j+1)$ or $\mathcal{L}(I_1,i)\cap\mathcal{L}(I_1,j)=\{I_1(i+1)\}$ but $I_1(i+1)=I_1(j)$ or $I_1(i+1)=I_1(j+1)$ or $\mathcal{L}(I_1,i)=\mathcal{L}(I_1,j)$.

Next we state a number of propositions:

(16) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that f is s.n.c. holds f is s.c.c..

(17) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that f is s.c.c. and $\mathcal{L}(f,1)$ misses $\mathcal{L}(f,\operatorname{len} f -' 1)$ holds f is s.n.c..

(18) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that f is nodic and PairF f is Simple holds f is s.c.c..

(19) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that f is nodic and PairF f is Simple and $f(1)\neq f(\operatorname{len} f)$ holds f is s.n.c..

(20) For all points p_1, p_2, p_3 of $\mathcal{E}^n_{\mathrm{T}}$ such that there exists a set x such that $x\neq p_2$ and $x\in\mathcal{L}(p_1,p_2)\cap\mathcal{L}(p_2,p_3)$ holds $p_1\in\mathcal{L}(p_2,p_3)$ or $p_3\in\mathcal{L}(p_1,p_2)$.

(21) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is s.n.c. and $\mathcal{L}(f,1)\cap\mathcal{L}(f,1+1)\subseteq\{f_{1+1}\}$ and $\mathcal{L}(f,\operatorname{len} f -' 2)\cap\mathcal{L}(f,\operatorname{len} f -' 1)\subseteq\{f_{\operatorname{len} f -' 1}\}$. Then f is unfolded.

(22) For every finite sequence f of elements of X such that PairF f is Simple and $f(1)\neq f(\operatorname{len} f)$ holds f is one-to-one and $\operatorname{len} f\neq 1$.

(23) For every finite sequence f of elements of X such that f is one-to-one and $\operatorname{len} f>1$ holds PairF f is Simple and $f(1)\neq f(\operatorname{len} f)$.

(24) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If f is nodic and PairF f is Simple and $f(1)\neq f(\operatorname{len} f)$, then f is unfolded.

(25) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$ and given i. Suppose g is Shortcut of f and $1\leq i$ and $i+1\leq\operatorname{len} g$. Then there exists a natural number k_1 such that $1\leq k_1$ and $k_1+1\leq\operatorname{len} f$ and $f_{k_1}=g_i$ and $f_{k_1+1}=g_{i+1}$ and $f(k_1)=g(i)$ and $f(k_1+1)=g(i+1)$.

(26) For all finite sequences f, g of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that g is Shortcut of f holds $\operatorname{rng} g\subseteq\operatorname{rng} f$.

(27) For all finite sequences f, g of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that g is Shortcut of f holds $\widetilde{\mathcal{L}}(g)\subseteq\widetilde{\mathcal{L}}(f)$.

(28) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. If f is special and g is Shortcut of f, then g is special.

(29) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is special and $2\leq\operatorname{len} f$ and $f(1)\neq f(\operatorname{len} f)$. Then there exists a finite sequence g of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $2\leq\operatorname{len} g$ and g is special and one-to-one and $\widetilde{\mathcal{L}}(g)\subseteq\widetilde{\mathcal{L}}(f)$ and $f(1)=g(1)$ and $f(\operatorname{len} f)=g(\operatorname{len} g)$ and $\operatorname{rng} g\subseteq\operatorname{rng} f$.

(30) Let f_1, f_4 be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f_1 is special and f_4 is special and $2\leq\operatorname{len} f_1$ and $2\leq\operatorname{len} f_4$ and $f_1(1)\neq f_1(\operatorname{len} f_1)$ and $f_4(1)\neq f_4(\operatorname{len} f_4)$ and **X**-coordinate(f_1) lies between (**X**-coordinate(f_1))(1) and (**X**-coordinate(f_1))($\operatorname{len} f_1$) and **X**-coordinate(f_4) lies between (**X**-coordinate(f_1))(1) and (**X**-coordinate(f_1))($\operatorname{len} f_1$) and **Y**-coordinate(f_1) lies between (**Y**-coordinate(f_4))(1) and (**Y**-coordinate(f_4))($\operatorname{len} f_4$) and **Y**-coordinate(f_4) lies between (**Y**-coordinate(f_4))(1) and (**Y**-coordinate(f_4))($\operatorname{len} f_4$). Then $\widetilde{\mathcal{L}}(f_1)$ meets $\widetilde{\mathcal{L}}(f_4)$.

3. Norm of Points in $\mathcal{E}^n_{\mathrm{T}}$

The following proposition is true

(31) For all real numbers a, b, r_1, r_2 such that $a \leq r_1$ and $r_1 \leq b$ and $a \leq r_2$ and $r_2 \leq b$ holds $|r_1 - r_2| \leq b - a$.

Let us consider n and let p be a point of $\mathcal{E}^n_{\mathrm{T}}$. Then $|p|$ can be characterized by the condition:

(Def. 5) For every element w of $\mathcal{R}^n$ such that $p = w$ holds $|p| = |w|$.

In the sequel p, p_1, p_2 denote points of $\mathcal{E}^n_{\mathrm{T}}$.
We now state several propositions:

(45)[2] For all points x_1, x_2 of $\mathcal{E}^n$ such that $x_1 = p_1$ and $x_2 = p_2$ holds $|p_1 - p_2| = \rho(x_1, x_2)$.

(46) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $|p|^{\mathbf{2}} = (p_{\mathbf{1}})^{\mathbf{2}} + (p_{\mathbf{2}})^{\mathbf{2}}$.

(47) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $|p| = \sqrt{(p_{\mathbf{1}})^{\mathbf{2}} + (p_{\mathbf{2}})^{\mathbf{2}}}$.

(48) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $|p| \leq |p_{\mathbf{1}}| + |p_{\mathbf{2}}|$.

(49) For all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ holds $|p_1 - p_2| \leq |(p_1)_{\mathbf{1}} - (p_2)_{\mathbf{1}}| + |(p_1)_{\mathbf{2}} - (p_2)_{\mathbf{2}}|$.

(50) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $|p_{\mathbf{1}}| \leq |p|$ and $|p_{\mathbf{2}}| \leq |p|$.

(51) For all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ holds $|(p_1)_{\mathbf{1}} - (p_2)_{\mathbf{1}}| \leq |p_1 - p_2|$ and $|(p_1)_{\mathbf{2}} - (p_2)_{\mathbf{2}}| \leq |p_1 - p_2|$.

(52) If $p \in \mathcal{L}(p_1, p_2)$, then there exists r such that $0 \leq r$ and $r \leq 1$ and $p = (1 - r) \cdot p_1 + r \cdot p_2$.

(53) If $p \in \mathcal{L}(p_1, p_2)$, then $|p - p_1| \leq |p_1 - p_2|$ and $|p - p_2| \leq |p_1 - p_2|$.

4. Extended Goboard Theorem and Fashoda Meet Theorem

In the sequel M denotes a non empty metric space.
Next we state several propositions:

(54) For all subsets P, Q of M_{top} such that $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact holds $\mathrm{dist}^{\min}_{\min}(P, Q) \geq 0$.

(55) Let P, Q be subsets of M_{top}. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then P misses Q if and only if $\mathrm{dist}^{\min}_{\min}(P, Q) > 0$.

(56) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and a, c, d be real numbers. Suppose that

(i) $1 \leq \mathrm{len}\, f$,

(ii) $\mathbf{X}$-coordinate(f) lies between $(\mathbf{X}$-coordinate$(f))(1)$ and $(\mathbf{X}$-coordinate$(f))(\mathrm{len}\, f)$,

(iii) $\mathbf{Y}$-coordinate(f) lies between c and d,

(iv) $a > 0$, and

(v) for every i such that $1 \leq i$ and $i + 1 \leq \mathrm{len}\, f$ holds $|f_i - f_{i+1}| < a$.

Then there exists a finite sequence g of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that

g is special and $g(1) = f(1)$ and $g(\mathrm{len}\, g) = f(\mathrm{len}\, f)$ and $\mathrm{len}\, g \geq \mathrm{len}\, f$ and $\mathbf{X}$-coordinate(g) lies between $(\mathbf{X}$-coordinate$(f))(1)$ and $(\mathbf{X}$-coordinate$(f))(\mathrm{len}\, f)$ and $\mathbf{Y}$-coordinate(g) lies between c and d and for every j such that $j \in \mathrm{dom}\, g$ there exists k such that $k \in \mathrm{dom}\, f$ and $|g_j - f_k| < a$ and for every j such that $1 \leq j$ and $j + 1 \leq \mathrm{len}\, g$ holds $|g_j - g_{j+1}| < a$.

[2] The propositions (32)–(44) have been removed.

(57) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and a, c, d be real numbers. Suppose that

(i) $1 \leq \operatorname{len} f$,

(ii) **Y**-coordinate(f) lies between $(\mathbf{Y}\text{-coordinate}(f))(1)$ and $(\mathbf{Y}\text{-coordinate}(f))(\operatorname{len} f)$,

(iii) **X**-coordinate(f) lies between c and d,

(iv) $a > 0$, and

(v) for every i such that $1 \leq i$ and $i+1 \leq \operatorname{len} f$ holds $|f_i - f_{i+1}| < a$.

Then there exists a finite sequence g of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that

g is special and $g(1) = f(1)$ and $g(\operatorname{len} g) = f(\operatorname{len} f)$ and $\operatorname{len} g \geq \operatorname{len} f$ and **Y**-coordinate(g) lies between $(\mathbf{Y}\text{-coordinate}(f))(1)$ and $(\mathbf{Y}\text{-coordinate}(f))(\operatorname{len} f)$ and **X**-coordinate(g) lies between c and d and for every j such that $j \in \operatorname{dom} g$ there exists k such that $k \in \operatorname{dom} f$ and $|g_j - f_k| < a$ and for every j such that $1 \leq j$ and $j+1 \leq \operatorname{len} g$ holds $|g_j - g_{j+1}| < a$.

(59)[3] For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $1 \leq \operatorname{len} f$ holds $\operatorname{len} \mathbf{X}\text{-coordinate}(f) = \operatorname{len} f$ and $(\mathbf{X}\text{-coordinate}(f))(1) = (f_1)_{\mathbf{1}}$ and $(\mathbf{X}\text{-coordinate}(f))(\operatorname{len} f) = (f_{\operatorname{len} f})_{\mathbf{1}}$.

(60) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $1 \leq \operatorname{len} f$ holds $\operatorname{len} \mathbf{Y}\text{-coordinate}(f) = \operatorname{len} f$ and $(\mathbf{Y}\text{-coordinate}(f))(1) = (f_1)_{\mathbf{2}}$ and $(\mathbf{Y}\text{-coordinate}(f))(\operatorname{len} f) = (f_{\operatorname{len} f})_{\mathbf{2}}$.

(61) For every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $i \in \operatorname{dom} f$ holds $(\mathbf{X}\text{-coordinate}(f))(i) = (f_i)_{\mathbf{1}}$ and $(\mathbf{Y}\text{-coordinate}(f))(i) = (f_i)_{\mathbf{2}}$.

(62) Let P, Q be non empty subsets of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) P is an arc from p_1 to p_2,

(ii) Q is an arc from q_1 to q_2,

(iii) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in P$ holds $(p_1)_{\mathbf{1}} \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$,

(iv) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in Q$ holds $(p_1)_{\mathbf{1}} \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$,

(v) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in P$ holds $(q_1)_{\mathbf{2}} \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq (q_2)_{\mathbf{2}}$, and

(vi) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in Q$ holds $(q_1)_{\mathbf{2}} \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq (q_2)_{\mathbf{2}}$.

Then P meets Q.

In the sequel X, Y are non empty topological spaces.
The following propositions are true:

(63) Let f be a map from X into Y, P be a non empty subset of Y, and f_1 be a map from X into $Y \restriction P$. If $f = f_1$ and f is continuous, then f_1 is continuous.

(64) Let f be a map from X into Y and P be a non empty subset of Y. Suppose X is compact and Y is a T_2 space and f is continuous and one-to-one and $P = \operatorname{rng} f$. Then there exists a map f_1 from X into $Y \restriction P$ such that $f = f_1$ and f_1 is a homeomorphism.

(65) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $f(O)_{\mathbf{1}} = a$ and $f(I)_{\mathbf{1}} = b$ and $g(O)_{\mathbf{2}} = c$ and $g(I)_{\mathbf{2}} = d$ and for every point r of $\mathbb{I}$ holds $a \leq f(r)_{\mathbf{1}}$ and $f(r)_{\mathbf{1}} \leq b$ and $a \leq g(r)_{\mathbf{1}}$ and $g(r)_{\mathbf{1}} \leq b$ and $c \leq f(r)_{\mathbf{2}}$ and $f(r)_{\mathbf{2}} \leq d$ and $c \leq g(r)_{\mathbf{2}}$ and $g(r)_{\mathbf{2}} \leq d$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

[3] The proposition (58) has been removed.

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Received August 21, 1998

Published January 2, 2004