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Integrability of Bounded Total Functions

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Summary. All these results have been obtained by Darboux's theorem in our previous article [10]. In addition, we have proved the first mean value theorem to Riemann integral.

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The articles [21], [24], [1], [22], [12], [3], [8], [25], [2], [15], [16], [6], [14], [13], [20], [19], [17], [23], [7], [9], [11], [18], [5], and [4] provide the notation and terminology for this paper.

1. Basic Integrable Functions and First Mean Value Theorem

For simplicity, we adopt the following convention: i, n denote natural numbers, a, r, x, y denote real numbers, A denotes a closed-interval subset of $\mathbb{R}$, C denotes a non empty set, and X denotes a set.

We now state several propositions:

(1) For every element D of $\operatorname{divs} A$ such that $\operatorname{vol}(A) = 0$ holds $\operatorname{len} D = 1$.

(2) $\chi_{A,A}$ is integrable on A and $\operatorname{integral} \chi_{A,A} = \operatorname{vol}(A)$.

(3) For every partial function f from A to $\mathbb{R}$ and for every r holds f is total and $\operatorname{rng} f = \{r\}$ iff $f = r\,\chi_{A,A}$.

(4) For every function f from A into $\mathbb{R}$ and for every r such that $\operatorname{rng} f = \{r\}$ holds f is integrable on A and $\operatorname{integral} f = r \cdot \operatorname{vol}(A)$.

(5) For every r there exists a function f from A into $\mathbb{R}$ such that $\operatorname{rng} f = \{r\}$ and f is bounded on A.

(6) Let f be a partial function from A to $\mathbb{R}$ and D be an element of $\operatorname{divs} A$. If $\operatorname{vol}(A) = 0$, then f is integrable on A and $\operatorname{integral} f = 0$.

(7) Let f be a function from A into $\mathbb{R}$. Suppose f is bounded on A and f is integrable on A. Then there exists a such that $\inf \operatorname{rng} f \leq a$ and $a \leq \sup \operatorname{rng} f$ and $\operatorname{integral} f = a \cdot \operatorname{vol}(A)$.

2. Integrability of Bounded Total Functions

One can prove the following propositions:

(8) Let f be a function from A into $\mathbb{R}$ and T be a DivSequence of A. Suppose f is bounded on A and δ_T is convergent and $\lim(\delta_T) = 0$. Then $\operatorname{lower\_sum}(f,T)$ is convergent and $\lim \operatorname{lower\_sum}(f,T) = \operatorname{lower\_integral} f$.

(9) Let f be a function from A into $\mathbb{R}$ and T be a DivSequence of A. Suppose f is bounded on A and δ_T is convergent and $\lim(\delta_T) = 0$. Then $\mathrm{upper\_sum}(f,T)$ is convergent and $\lim \mathrm{upper\_sum}(f,T) = \mathrm{upper\_integral}\, f$.

(10) Let f be a function from A into $\mathbb{R}$. Suppose f is bounded on A. Then f is upper integrable on A and f is lower integrable on A.

Let A be a closed-interval subset of $\mathbb{R}$, let I_1 be an element of $\mathrm{divs}\,A$, and let us consider n. We say that I_1 divides into equal n if and only if:

(Def. 1) $\mathrm{len}\, I_1 = n$ and for every i such that $i \in \mathrm{dom}\, I_1$ holds $I_1(i) = \inf A + \frac{\mathrm{vol}(A)}{\mathrm{len}\, I_1} \cdot i$.

We now state a number of propositions:

(11) There exists a DivSequence T of A such that δ_T is convergent and $\lim(\delta_T) = 0$.

(12) Let f be a function from A into $\mathbb{R}$. Suppose f is bounded on A. Then f is integrable on A if and only if for every DivSequence T of A such that δ_T is convergent and $\lim(\delta_T) = 0$ holds $\lim \mathrm{upper\_sum}(f,T) - \lim \mathrm{lower\_sum}(f,T) = 0$.

(13) For every function f from C into $\mathbb{R}$ holds $\max_+(f)$ is total and $\max_-(f)$ is total.

(14) For every partial function f from C to $\mathbb{R}$ such that f is upper bounded on X holds $\max_+(f)$ is upper bounded on X.

(15) For every partial function f from C to $\mathbb{R}$ holds $\max_+(f)$ is lower bounded on X.

(16) For every partial function f from C to $\mathbb{R}$ such that f is lower bounded on X holds $\max_-(f)$ is upper bounded on X.

(17) For every partial function f from C to $\mathbb{R}$ holds $\max_-(f)$ is lower bounded on X.

(18) For every partial function f from A to $\mathbb{R}$ such that f is upper bounded on A holds $\mathrm{rng}(f \restriction X)$ is upper bounded.

(19) For every partial function f from A to $\mathbb{R}$ such that f is lower bounded on A holds $\mathrm{rng}(f \restriction X)$ is lower bounded.

(20) Let f be a function from A into $\mathbb{R}$. Suppose f is bounded on A and f is integrable on A. Then $\max_+(f)$ is integrable on A.

(21) For every partial function f from C to $\mathbb{R}$ holds $\max_-(f) = \max_+(-f)$.

(22) Let f be a function from A into $\mathbb{R}$. Suppose f is bounded on A and f is integrable on A. Then $\max_-(f)$ is integrable on A.

(23) Let f be a function from A into $\mathbb{R}$. Suppose f is bounded on A and f is integrable on A. Then $|f|$ is integrable on A and $|\mathrm{integral}\, f| \leq \mathrm{integral}|f|$.

(24) For every function f from A into $\mathbb{R}$ such that for all x, y such that $x \in A$ and $y \in A$ holds $|f(x) - f(y)| \leq a$ holds $\sup \mathrm{rng}\, f - \inf \mathrm{rng}\, f \leq a$.

(25) Let f, g be functions from A into $\mathbb{R}$. Suppose f is bounded on A and $a \geq 0$ and for all x, y such that $x \in A$ and $y \in A$ holds $|g(x) - g(y)| \leq a \cdot |f(x) - f(y)|$. Then $\sup \mathrm{rng}\, g - \inf \mathrm{rng}\, g \leq a \cdot (\sup \mathrm{rng}\, f - \inf \mathrm{rng}\, f)$.

(26) Let f, g, h be functions from A into $\mathbb{R}$. Suppose that

(i) f is bounded on A,

(ii) g is bounded on A,

(iii) $a \geq 0$, and

(iv) for all x, y such that $x \in A$ and $y \in A$ holds $|h(x) - h(y)| \leq a \cdot (|f(x) - f(y)| + |g(x) - g(y)|)$.

Then $\sup \mathrm{rng}\, h - \inf \mathrm{rng}\, h \leq a \cdot ((\sup \mathrm{rng}\, f - \inf \mathrm{rng}\, f) + (\sup \mathrm{rng}\, g - \inf \mathrm{rng}\, g))$.

(27) Let f, g be functions from A into $\mathbb{R}$. Suppose that

(i) f is bounded on A,

(ii) f is integrable on A,

(iii) g is bounded on A,

(iv) $a > 0$, and

(v) for all x, y such that $x \in A$ and $y \in A$ holds $|g(x) - g(y)| \leq a \cdot |f(x) - f(y)|$.

Then g is integrable on A.

(28) Let f, g, h be functions from A into $\mathbb{R}$. Suppose that f is bounded on A and f is integrable on A and g is bounded on A and g is integrable on A and h is bounded on A and $a > 0$ and for all x, y such that $x \in A$ and $y \in A$ holds $|h(x) - h(y)| \leq a \cdot (|f(x) - f(y)| + |g(x) - g(y)|)$. Then h is integrable on A.

(29) Let f, g be functions from A into $\mathbb{R}$. Suppose f is bounded on A and f is integrable on A and g is bounded on A and g is integrable on A. Then $f\,g$ is integrable on A.

(30) Let f be a function from A into $\mathbb{R}$. Suppose f is bounded on A and f is integrable on A and $0 \notin \operatorname{rng} f$ and $\frac{1}{f}$ is bounded on A. Then $\frac{1}{f}$ is integrable on A.

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