JOURNAL OF FORMALIZED MATHEMATICS
Volume 11, Released 1999, Published 2003
Inst. of Computer Science, Univ. of Białystok

Darboux's Theorem

Noboru Endou
Shinshu University
Nagano

Katsumi Wasaki
Shinshu University
Nagano

Yasunari Shidama
Shinshu University
Nagano

Summary. In this article, we have proved the Darboux's theorem. This theorem is important to prove the Riemann integrability. We can replace an upper bound and a lower bound of a function which is the definition of Riemann integration with convergence of sequence by Darboux's theorem.

MML Identifier: `INTEGRA3`.

WWW: `http://mizar.org/JFM/Vol11/integra3.html`

The articles [20], [22], [2], [21], [11], [23], [4], [5], [24], [12], [6], [8], [3], [15], [7], [9], [14], [13], [17], [19], [18], [16], [1], and [10] provide the notation and terminology for this paper.

1. Lemmas of Division

We adopt the following rules: x, y are real numbers, i, j, k are natural numbers, and p, q are finite sequences of elements of $\mathbb{R}$.

We now state a number of propositions:

(1) Let A be a closed-interval subset of $\mathbb{R}$ and D be an element of $\mathrm{divs}A$. If $\mathrm{vol}(A) \neq 0$, then there exists i such that $i \in \mathrm{dom}D$ and $\mathrm{vol}(\mathrm{divset}(D,i)) > 0$.

(2) Let A be a closed-interval subset of $\mathbb{R}$ and D be an element of $\mathrm{divs}A$. If $x \in A$, then there exists j such that $j \in \mathrm{dom}D$ and $x \in \mathrm{divset}(D,j)$.

(3) Let A be a closed-interval subset of $\mathbb{R}$ and D_1, D_2 be elements of $\mathrm{divs}A$. Then there exists an element D of $\mathrm{divs}A$ such that $D_1 \leq D$ and $D_2 \leq D$ and $\mathrm{rng}D = \mathrm{rng}D_1 \cup \mathrm{rng}D_2$.

(4) Let A be a closed-interval subset of $\mathbb{R}$ and D, D_1 be elements of $\mathrm{divs}A$. Suppose $\delta_{(D_1)} < \mathrm{min}\,\mathrm{rng}\,\mathrm{upper\_volume}(\chi_{A,A},D)$. Let given x, y, i. If $i \in \mathrm{dom}D_1$ and $x \in \mathrm{rng}D \cap \mathrm{divset}(D_1,i)$ and $y \in \mathrm{rng}D \cap \mathrm{divset}(D_1,i)$, then $x = y$.

(5) For all p, q such that $\mathrm{rng}\,p = \mathrm{rng}\,q$ and p is increasing and q is increasing holds $p = q$.

(6) Let A be a closed-interval subset of $\mathbb{R}$ and D, D_1 be elements of $\mathrm{divs}A$. If $D \leq D_1$ and $i \in \mathrm{dom}D$ and $j \in \mathrm{dom}D$ and $i \leq j$, then $\mathrm{indx}(D_1,D,i) \leq \mathrm{indx}(D_1,D,j)$ and $\mathrm{indx}(D_1,D,i) \in \mathrm{dom}D_1$.

(7) Let A be a closed-interval subset of $\mathbb{R}$ and D, D_1 be elements of $\mathrm{divs}A$. If $D \leq D_1$ and $i \in \mathrm{dom}D$ and $j \in \mathrm{dom}D$ and $i < j$, then $\mathrm{indx}(D_1,D,i) < \mathrm{indx}(D_1,D,j)$.

(8) For every closed-interval subset A of $\mathbb{R}$ and for every element D of $\mathrm{divs}A$ holds $\delta_D \geq 0$.

(9) Let A be a closed-interval subset of $\mathbb{R}$, g be a function from A into $\mathbb{R}$, and D_1, D_2 be elements of divs$A$. Suppose $x \in \operatorname{divset}(D_1, \operatorname{len} D_1)$ and $\operatorname{len} D_1 \geq 2$ and $D_1 \leq D_2$ and $\operatorname{rng} D_2 = \operatorname{rng} D_1 \cup \{x\}$ and g is bounded on A. Then $\sum \text{lower\_volume}(g, D_2) - \sum \text{lower\_volume}(g, D_1) \leq (\sup \operatorname{rng} g - \inf \operatorname{rng} g) \cdot \delta_{(D_1)}$.

(10) Let A be a closed-interval subset of $\mathbb{R}$, g be a function from A into $\mathbb{R}$, and D_1, D_2 be elements of divs$A$. Suppose $x \in \operatorname{divset}(D_1, \operatorname{len} D_1)$ and $\operatorname{len} D_1 \geq 2$ and $D_1 \leq D_2$ and $\operatorname{rng} D_2 = \operatorname{rng} D_1 \cup \{x\}$ and g is bounded on A. Then $\sum \text{upper\_volume}(g, D_1) - \sum \text{upper\_volume}(g, D_2) \leq (\sup \operatorname{rng} g - \inf \operatorname{rng} g) \cdot \delta_{(D_1)}$.

(11) Let A be a closed-interval subset of $\mathbb{R}$, D be an element of divs$A$, r be a real number, and i, j be natural numbers. Suppose $i \in \operatorname{dom} D$ and $j \in \operatorname{dom} D$ and $i \leq j$ and $r < (\operatorname{mid}(D, i, j))(1)$. Then there exists a closed-interval subset B of $\mathbb{R}$ such that $r = \inf B$ and $\sup B = (\operatorname{mid}(D, i, j))(\operatorname{len} \operatorname{mid}(D, i, j))$ and $\operatorname{len} \operatorname{mid}(D, i, j) = (j - i) + 1$ and $\operatorname{mid}(D, i, j)$ is a DivisionPoint of B.

(12) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and D_1, D_2 be elements of divs$A$. Suppose $x \in \operatorname{divset}(D_1, \operatorname{len} D_1)$ and $\operatorname{vol}(A) \neq 0$ and $D_1 \leq D_2$ and $\operatorname{rng} D_2 = \operatorname{rng} D_1 \cup \{x\}$ and f is bounded on A and $x > \inf A$. Then $\sum \text{lower\_volume}(f, D_2) - \sum \text{lower\_volume}(f, D_1) \leq (\sup \operatorname{rng} f - \inf \operatorname{rng} f) \cdot \delta_{(D_1)}$.

(13) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and D_1, D_2 be elements of divs$A$. Suppose $x \in \operatorname{divset}(D_1, \operatorname{len} D_1)$ and $\operatorname{vol}(A) \neq 0$ and $D_1 \leq D_2$ and $\operatorname{rng} D_2 = \operatorname{rng} D_1 \cup \{x\}$ and f is bounded on A and $x > \inf A$. Then $\sum \text{upper\_volume}(f, D_1) - \sum \text{upper\_volume}(f, D_2) \leq (\sup \operatorname{rng} f - \inf \operatorname{rng} f) \cdot \delta_{(D_1)}$.

(14) Let A be a closed-interval subset of $\mathbb{R}$, D_1, D_2 be elements of divs$A$, r be a real number, and i, j be natural numbers. Suppose $i \in \operatorname{dom} D_1$ and $j \in \operatorname{dom} D_1$ and $i \leq j$ and $D_1 \leq D_2$ and $r < (\operatorname{mid}(D_2, \operatorname{indx}(D_2, D_1, i), \operatorname{indx}(D_2, D_1, j)))(1)$. Then there exists a closed-interval subset B of $\mathbb{R}$ and there exist elements M_1, M_2 of divs$B$ such that $r = \inf B$ and $\sup B = M_2(\operatorname{len} M_2)$ and $\sup B = M_1(\operatorname{len} M_1)$ and $M_1 \leq M_2$ and $M_1 = \operatorname{mid}(D_1, i, j)$ and $M_2 = \operatorname{mid}(D_2, \operatorname{indx}(D_2, D_1, i), \operatorname{indx}(D_2, D_1, j))$.

(15) For every closed-interval subset A of $\mathbb{R}$ and for every element D of divs$A$ such that $x \in \operatorname{rng} D$ holds $D(1) \leq x$ and $x \leq D(\operatorname{len} D)$.

(16) Let p be a finite sequence of elements of $\mathbb{R}$ and given i, j, k. Suppose p is increasing and $i \in \operatorname{dom} p$ and $j \in \operatorname{dom} p$ and $k \in \operatorname{dom} p$ and $p(i) \leq p(k)$ and $p(k) \leq p(j)$. Then $p(k) \in \operatorname{rng} \operatorname{mid}(p, i, j)$.

(17) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and D be an element of divs$A$. If f is bounded on A and $i \in \operatorname{dom} D$, then $\inf \operatorname{rng}(f \restriction \operatorname{divset}(D, i)) \leq \sup \operatorname{rng} f$.

(18) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and D be an element of divs$A$. If f is bounded on A and $i \in \operatorname{dom} D$, then $\sup \operatorname{rng}(f \restriction \operatorname{divset}(D, i)) \geq \inf \operatorname{rng} f$.

2. Darboux's Theorem

Next we state two propositions:

(19) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and T be a DivSequence of A. Suppose f is bounded on A and δ_T is convergent to 0 and $\operatorname{vol}(A) \neq 0$. Then $\text{lower\_sum}(f, T)$ is convergent and $\lim \text{lower\_sum}(f, T) = \text{lower\_integral} f$.

(20) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and T be a DivSequence of A. Suppose f is bounded on A and δ_T is convergent to 0 and $\operatorname{vol}(A) \neq 0$. Then $\text{upper\_sum}(f, T)$ is convergent and $\lim \text{upper\_sum}(f, T) = \text{upper\_integral} f$.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[6] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[7] Czesław Byliński. The sum and product of finite sequences of real numbers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rvsum_1.html`.

[8] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Journal of Formalized Mathematics*, 9, 1997. `http://mizar.org/JFM/Vol9/pscomp_1.html`.

[9] Noboru Endou and Artur Korniłowicz. The definition of the Riemann definite integral and some related lemmas. *Journal of Formalized Mathematics*, 11, 1999. `http://mizar.org/JFM/Vol11/integra1.html`.

[10] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Scalar multiple of Riemann definite integral. *Journal of Formalized Mathematics*, 11, 1999. `http://mizar.org/JFM/Vol11/integra2.html`.

[11] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[12] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seq_4.html`.

[13] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seq_2.html`.

[14] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seq_1.html`.

[15] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rfunct_1.html`.

[16] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-Board — part I. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/goboard1.html`.

[17] Yatsuka Nakamura and Roman Matuszewski. Reconstructions of special sequences. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/jordan3.html`.

[18] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/fdiff_1.html`.

[19] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rcomp_1.html`.

[20] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[21] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[22] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[23] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[24] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

Received December 7, 1999

Published January 2, 2004