JOURNAL OF FORMALIZED MATHEMATICS
Volume 11, Released 1999, Published 2003
Inst. of Computer Science, Univ. of Białystok

Scalar Multiple of Riemann Definite Integral

Noboru Endou
Shinshu University
Nagano

Katsumi Wasaki
Shinshu University
Nagano

Yasunari Shidama
Shinshu University
Nagano

Summary. The goal of this article is to prove a scalar multiplicity of Riemann definite integral. Therefore, we defined a scalar product to the subset of real space, and we proved some relating lemmas. At last, we proved a scalar multiplicity of Riemann definite integral. As a result, a linearity of Riemann definite integral was proven by unifying the previous article [11].

MML Identifier: `INTEGRA2`.

WWW: `http://mizar.org/JFM/Vol11/integra2.html`

The articles [20], [22], [3], [21], [12], [2], [5], [23], [13], [7], [6], [9], [4], [15], [8], [11], [14], [17], [18], [19], [10], [1], and [16] provide the notation and terminology for this paper.

1. Lemmas of Finite Sequence

We follow the rules: r, x, y are real numbers, i, j are natural numbers, and p is a finite sequence of elements of $\mathbb{R}$.

Next we state the proposition

(1) For every closed-interval subset A of $\mathbb{R}$ and for every real number x holds $x \in A$ iff $\inf A \leq x$ and $x \leq \sup A$.

Let I_1 be a finite sequence of elements of $\mathbb{R}$. We say that I_1 is non-decreasing if and only if:

(Def. 1) For every natural number n such that $n \in \mathrm{dom}\, I_1$ and $n+1 \in \mathrm{dom}\, I_1$ holds $I_1(n) \leq I_1(n+1)$.

Let us note that there exists a finite sequence of elements of $\mathbb{R}$ which is non-decreasing.

We now state three propositions:

(2) Let p be a non-decreasing finite sequence of elements of $\mathbb{R}$ and given i, j. If $i \in \mathrm{dom}\, p$ and $j \in \mathrm{dom}\, p$ and $i \leq j$, then $p(i) \leq p(j)$.

(3) Let given p. Then there exists a non-decreasing finite sequence q of elements of $\mathbb{R}$ such that p and q are fiberwise equipotent.

(4) Let D be a non empty set, f be a finite sequence of elements of D, and k_1, k_2, k_3 be natural numbers. If $1 \leq k_1$ and $k_3 \leq \mathrm{len}\, f$ and $k_1 \leq k_2$ and $k_2 < k_3$, then $(\mathrm{mid}(f, k_1, k_2)) ^\frown \mathrm{mid}(f, k_2 + 1, k_3) = \mathrm{mid}(f, k_1, k_3)$.

2. Scalar Product of Real Subset

Let A be a subset of $\mathbb{R}$ and let x be a real number. The functor $x \cdot A$ yields a subset of $\mathbb{R}$ and is defined by:

(Def. 2) For every real number y holds $y \in x \cdot A$ iff there exists a real number z such that $z \in A$ and $y = x \cdot z$.

The following propositions are true:

(5) Let X, Y be non empty sets and f be a partial function from X to $\mathbb{R}$. If f is upper bounded on X and $Y \subseteq X$, then $f \restriction Y$ is upper bounded on Y.

(6) Let X, Y be non empty sets and f be a partial function from X to $\mathbb{R}$. If f is lower bounded on X and $Y \subseteq X$, then $f \restriction Y$ is lower bounded on Y.

(7) For every non empty subset X of $\mathbb{R}$ holds $r \cdot X$ is non empty.

(8) For every subset X of $\mathbb{R}$ holds $r \cdot X = \{r \cdot x : x \in X\}$.

(9) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded and $0 \leq r$ holds $r \cdot X$ is upper bounded.

(10) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded and $r \leq 0$ holds $r \cdot X$ is lower bounded.

(11) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded and $0 \leq r$ holds $r \cdot X$ is lower bounded.

(12) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded and $r \leq 0$ holds $r \cdot X$ is upper bounded.

(13) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded and $0 \leq r$ holds $\sup(r \cdot X) = r \cdot \sup X$.

(14) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded and $r \leq 0$ holds $\inf(r \cdot X) = r \cdot \sup X$.

(15) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded and $0 \leq r$ holds $\inf(r \cdot X) = r \cdot \inf X$.

(16) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded and $r \leq 0$ holds $\sup(r \cdot X) = r \cdot \inf X$.

3. Scalar Multiple of Integral

One can prove the following propositions:

(17) For every non empty set X and for every function f from X into $\mathbb{R}$ holds $\mathrm{rng}(r\, f) = r \cdot \mathrm{rng}\, f$.

(18) For all non empty sets X, Z and for every partial function f from X to $\mathbb{R}$ holds $\mathrm{rng}(r\,(f \restriction Z)) = r \cdot \mathrm{rng}(f \restriction Z)$.

(19) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and D be an element of $\mathrm{divs}\, A$. If f is bounded on A and $r \geq 0$, then $(\mathrm{upper\_sum\_set}\, r\, f)(D) \geq r \cdot \inf \mathrm{rng}\, f \cdot \mathrm{vol}(A)$.

(20) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and D be an element of $\mathrm{divs}\, A$. If f is bounded on A and $r \leq 0$, then $(\mathrm{upper\_sum\_set}\, r\, f)(D) \geq r \cdot \sup \mathrm{rng}\, f \cdot \mathrm{vol}(A)$.

(21) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and D be an element of $\mathrm{divs}\, A$. If f is bounded on A and $r \geq 0$, then $(\mathrm{lower\_sum\_set}\, r\, f)(D) \leq r \cdot \sup \mathrm{rng}\, f \cdot \mathrm{vol}(A)$.

(22) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, and D be an element of $\operatorname{divs} A$. If f is bounded on A and $r \leq 0$, then $(\text{lower\_sum\_set}\, r\, f)(D) \leq r \cdot \inf \operatorname{rng} f \cdot \operatorname{vol}(A)$.

(23) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and given i. If $i \in \operatorname{Seg} \operatorname{len} D$ and f is upper bounded on A and $r \geq 0$, then $(\text{upper\_volume}(r\, f, D))(i) = r \cdot (\text{upper\_volume}(f, D))(i)$.

(24) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and given i. If $i \in \operatorname{Seg} \operatorname{len} D$ and f is upper bounded on A and $r \leq 0$, then $(\text{lower\_volume}(r\, f, D))(i) = r \cdot (\text{upper\_volume}(f, D))(i)$.

(25) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and given i. If $i \in \operatorname{Seg} \operatorname{len} D$ and f is lower bounded on A and $r \geq 0$, then $(\text{lower\_volume}(r\, f, D))(i) = r \cdot (\text{lower\_volume}(f, D))(i)$.

(26) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and given i. If $i \in \operatorname{Seg} \operatorname{len} D$ and f is lower bounded on A and $r \leq 0$, then $(\text{upper\_volume}(r\, f, D))(i) = r \cdot (\text{lower\_volume}(f, D))(i)$.

(27) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is upper bounded on A and $r \geq 0$, then $\text{upper\_sum}(r\, f, D) = r \cdot \text{upper\_sum}(f, D)$.

(28) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is upper bounded on A and $r \leq 0$, then $\text{lower\_sum}(r\, f, D) = r \cdot \text{upper\_sum}(f, D)$.

(29) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is lower bounded on A and $r \geq 0$, then $\text{lower\_sum}(r\, f, D) = r \cdot \text{lower\_sum}(f, D)$.

(30) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is lower bounded on A and $r \leq 0$, then $\text{upper\_sum}(r\, f, D) = r \cdot \text{lower\_sum}(f, D)$.

(31) Let A be a closed-interval subset of $\mathbb{R}$ and f be a function from A into $\mathbb{R}$. Suppose f is bounded on A and f is integrable on A. Then $r\, f$ is integrable on A and $\operatorname{integral} r\, f = r \cdot \operatorname{integral} f$.

4. Monotoneity of Integral

We now state three propositions:

(32) Let A be a closed-interval subset of $\mathbb{R}$ and f be a function from A into $\mathbb{R}$. Suppose f is bounded on A and f is integrable on A and for every x such that $x \in A$ holds $f(x) \geq 0$. Then $\operatorname{integral} f \geq 0$.

(33) Let A be a closed-interval subset of $\mathbb{R}$ and f, g be functions from A into $\mathbb{R}$. Suppose f is bounded on A and f is integrable on A and g is bounded on A and g is integrable on A. Then $f - g$ is integrable on A and $\operatorname{integral} f - g = \operatorname{integral} f - \operatorname{integral} g$.

(34) Let A be a closed-interval subset of $\mathbb{R}$ and f, g be functions from A into $\mathbb{R}$. Suppose that

(i) f is bounded on A,

(ii) f is integrable on A,

(iii) g is bounded on A,

(iv) g is integrable on A, and

(v) for every x such that $x \in A$ holds $f(x) \geq g(x)$.

Then $\operatorname{integral} f \geq \operatorname{integral} g$.

5. Definition of Division Sequence

We now state two propositions:

(35) Let A be a closed-interval subset of $\mathbb{R}$ and f be a function from A into $\mathbb{R}$. If f is bounded on A, then $\mathrm{rng}\,\mathrm{upper\_sum\_set}\,f$ is lower bounded.

(36) Let A be a closed-interval subset of $\mathbb{R}$ and f be a function from A into $\mathbb{R}$. If f is bounded on A, then $\mathrm{rng}\,\mathrm{lower\_sum\_set}\,f$ is upper bounded.

Let A be a closed-interval subset of $\mathbb{R}$. A DivSequence of A is a function from $\mathbb{N}$ into $\mathrm{divs}A$.

Let A be a closed-interval subset of $\mathbb{R}$ and let T be a DivSequence of A. The functor δ_T yielding a sequence of real numbers is defined by:

(Def. 3) For every i holds $\delta_T(i) = \delta_{T(i)}$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a partial function from A to $\mathbb{R}$, and let T be a DivSequence of A. The functor $\mathrm{upper\_sum}(f,T)$ yields a sequence of real numbers and is defined as follows:

(Def. 4) For every i holds $(\mathrm{upper\_sum}(f,T))(i) = \mathrm{upper\_sum}(f,T(i))$.

The functor $\mathrm{lower\_sum}(f,T)$ yielding a sequence of real numbers is defined as follows:

(Def. 5) For every i holds $(\mathrm{lower\_sum}(f,T))(i) = \mathrm{lower\_sum}(f,T(i))$.

One can prove the following propositions:

(37) Let A be a closed-interval subset of $\mathbb{R}$ and D_1, D_2 be elements of $\mathrm{divs}A$. If $D_1 \leq D_2$, then for every j such that $j \in \mathrm{dom}D_2$ there exists i such that $i \in \mathrm{dom}D_1$ and $\mathrm{divset}(D_2, j) \subseteq \mathrm{divset}(D_1,i)$.

(38) For all finite non empty subsets X, Y of $\mathbb{R}$ such that $X \subseteq Y$ holds $\max X \leq \max Y$.

(39) For all finite non empty subsets X, Y of $\mathbb{R}$ such that there exists y such that $y \in Y$ and $\max X \leq y$ holds $\max X \leq \max Y$.

(40) For all closed-interval subsets A, B of $\mathbb{R}$ such that $A \subseteq B$ holds $\mathrm{vol}(A) \leq \mathrm{vol}(B)$.

(41) For every closed-interval subset A of $\mathbb{R}$ and for all elements D_1, D_2 of $\mathrm{divs}A$ such that $D_1 \leq D_2$ holds $\delta_{(D_1)} \geq \delta_{(D_2)}$.

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Received December 7, 1999

Published January 2, 2004