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The Canonical Formulae

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The articles [18], [9], [25], [23], [24], [26], [6], [4], [2], [17], [8], [7], [5], [19], [10], [1], [3], [13], [14], [20], [21], [15], [12], [16], [11], and [22] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following three propositions:

(1) For every integer i holds i is even iff $i-1$ is odd.

(2) For every integer i holds i is odd iff $i-1$ is even.

(3) Let X be a trivial set and x be a set. Suppose $x \in X$. Let f be a function from X into X. Then x is a fixpoint of f.

Let A, B, C be sets. Observe that every function from A into C^B is function yielding.

The following three propositions are true:

(4) For every function yielding function f holds $\mathrm{Sub_f}\,\mathrm{rng}\, f = \mathrm{rng}\, f$.

(5) For all sets A, B, x and for every function f such that $x \in A$ and $f \in B^A$ holds $f(x) \in B$.

(6) For all sets A, B, C such that if $C = \emptyset$, then $B = \emptyset$ or $A = \emptyset$ and for every function f from A into C^B holds $\mathrm{dom}_\kappa f(\kappa) = A \longmapsto B$.

Let us mention that $\emptyset$ is function yielding.

In the sequel n denotes a natural number and p, q, r denote elements of HP-WFF.

The following proposition is true

(7) For every set x holds $\emptyset(x) = \emptyset$.

Let A be a set and let B be a functional set. One can verify that every function from A into B is function yielding.

One can prove the following propositions:

(8) For every set X and for every subset A of X holds $[0 \longmapsto 1, 1 \longmapsto 0] \cdot \chi_{A,X} = \chi_{A^c,X}$.

(9) For every set X and for every subset A of X holds $[0 \longmapsto 1, 1 \longmapsto 0] \cdot \chi_{A^c,X} = \chi_{A,X}$.

(10) For all sets a, b, x, y, x', y' such that $a \neq b$ and $[a \longmapsto x, b \longmapsto y] = [a \longmapsto x', b \longmapsto y']$ holds $x = x'$ and $y = y'$.

(11) For all sets a, b, x, y, X, Y such that $a \neq b$ and $x \in X$ and $y \in Y$ holds $[a \longmapsto x, b \longmapsto y] \in \prod[a \longmapsto X, b \longmapsto Y]$.

(12) For every non empty set D and for every function f from 2 into D there exist elements d_1, d_2 of D such that $f = [0 \longmapsto d_1, 1 \longmapsto d_2]$.

(13) For all sets a, b, c, d such that $a \neq b$ holds $[a \longmapsto c, b \longmapsto d] \cdot [a \longmapsto b, b \longmapsto a] = [a \longmapsto d, b \longmapsto c]$.

(14) For all sets a, b, c, d and for every function f such that $a \neq b$ and $c \in \mathrm{dom} f$ and $d \in \mathrm{dom} f$ holds $f \cdot [a \longmapsto c, b \longmapsto d] = [a \longmapsto f(c), b \longmapsto f(d)]$.

2. The Cartesian Product of Functions and the Frege Function

Let f, g be one-to-one functions. Note that $[:f, g:]$ is one-to-one.

We now state several propositions:

(15) Let A, B be non empty sets, C, D be sets, f be a function from C into A, and g be a function from D into B. Then $\pi_1(A \times B) \cdot [:f, g:] = f \cdot \pi_1(C \times D)$.

(16) Let A, B be non empty sets, C, D be sets, f be a function from C into A, and g be a function from D into B. Then $\pi_2(A \times B) \cdot [:f, g:] = g \cdot \pi_2(C \times D)$.

(17) For every function g holds $\emptyset \hookleftarrow g = \emptyset$.

(18) For every function yielding function f and for all functions g, h holds $f \hookleftarrow g \cdot h = (f \cdot h) \hookleftarrow (g \cdot h)$.

(19) Let C be a set, A be a non empty set, f be a function from A into $C^{(\emptyset \text{ qua set})}$, and g be a function from A into $\emptyset$. Then $\mathrm{rng}(f \hookleftarrow g) = \{\emptyset\}$.

(20) Let A, B, C be sets such that if $B = \emptyset$, then $A = \emptyset$. Let f be a function from A into C^B and g be a function from A into B. Then $\mathrm{rng}(f \hookleftarrow g) \subseteq C$.

(21) For all sets A, B, C such that if $C = \emptyset$, then $B = \emptyset$ or $A = \emptyset$ and for every function f from A into C^B holds $\mathrm{dom}\,\mathrm{Frege}(f) = B^A$.

(23)[1] For all sets A, B, C such that if $C = \emptyset$, then $B = \emptyset$ or $A = \emptyset$ and for every function f from A into C^B holds $\mathrm{rng}\,\mathrm{Frege}(f) \subseteq C^A$.

(24) Let A, B, C be sets such that if $C = \emptyset$, then $B = \emptyset$ or $A = \emptyset$. Let f be a function from A into C^B. Then $\mathrm{Frege}(f)$ is a function from B^A into C^A.

3. About Permutations

The following proposition is true

(25) For all sets A, B and for every permutation P of A and for every permutation Q of B holds $[:P, Q:]$ is bijective.

Let A, B be non empty sets, let P be a permutation of A, and let Q be a function from B into B. The functor $P \Rightarrow Q$ yields a function from B^A into B^A and is defined by:

(Def. 1) For every function f from A into B holds $(P \Rightarrow Q)(f) = Q \cdot f \cdot P^{-1}$.

Let A, B be non empty sets, let P be a permutation of A, and let Q be a permutation of B. One can verify that $P \Rightarrow Q$ is bijective.

We now state three propositions:

[1] The proposition (22) has been removed.

(26) Let A, B be non empty sets, P be a permutation of A, Q be a permutation of B, and f be a function from A into B. Then $(P \Rightarrow Q)^{-1}(f) = Q^{-1} \cdot f \cdot P$.

(27) For all non empty sets A, B and for every permutation P of A and for every permutation Q of B holds $(P \Rightarrow Q)^{-1} = P^{-1} \Rightarrow Q^{-1}$.

(28) Let A, B, C be non empty sets, f be a function from A into C^B, g be a function from A into B, P be a permutation of B, and Q be a permutation of C. Then $((P \Rightarrow Q) \cdot f) \hookleftarrow (P \cdot g) = Q \cdot f \hookleftarrow g$.

4. Set Valuations

A SetValuation is a non-empty many sorted set indexed by $\mathbb{N}$.

In the sequel V is a SetValuation.

Let us consider V. The functor $\mathrm{SetVal}\,V$ yields a many sorted set indexed by HP-WFF and is defined by the conditions (Def. 2).

(Def. 2)(i) $(\mathrm{SetVal}\,V)(\mathrm{VERUM}) = 1$,

(ii) for every n holds $(\mathrm{SetVal}\,V)(\mathrm{prop}\,n) = V(n)$, and

(iii) for all p, q holds $(\mathrm{SetVal}\,V)(p \wedge q) = [:(\mathrm{SetVal}\,V)(p), (\mathrm{SetVal}\,V)(q):]$ and $(\mathrm{SetVal}\,V)(p \Rightarrow q) = (\mathrm{SetVal}\,V)(q)^{(\mathrm{SetVal}\,V)(p)}$.

Let us consider V, p. The functor $\mathrm{SetVal}(V,p)$ is defined by:

(Def. 3) $\mathrm{SetVal}(V,p) = (\mathrm{SetVal}\,V)(p)$.

Let us consider V, p. One can check that $\mathrm{SetVal}(V,p)$ is non empty.

One can prove the following propositions:

(29) $\mathrm{SetVal}(V,\mathrm{VERUM}) = 1$.

(30) $\mathrm{SetVal}(V,\mathrm{prop}\,n) = V(n)$.

(31) $\mathrm{SetVal}(V,p \wedge q) = [:\mathrm{SetVal}(V,p), \mathrm{SetVal}(V,q):]$.

(32) $\mathrm{SetVal}(V,p \Rightarrow q) = (\mathrm{SetVal}(V,q))^{\mathrm{SetVal}(V,p)}$.

Let us consider V, p, q. Observe that $\mathrm{SetVal}(V,p \Rightarrow q)$ is functional.

Let us consider V, p, q, r. One can check that every element of $\mathrm{SetVal}(V,p \Rightarrow (q \Rightarrow r))$ is function yielding.

Let us consider V, p, q, r. One can check that there exists a function from $\mathrm{SetVal}(V,p \Rightarrow q)$ into $\mathrm{SetVal}(V,p \Rightarrow r)$ which is function yielding and there exists an element of $\mathrm{SetVal}(V,p \Rightarrow (q \Rightarrow r))$ which is function yielding.

5. Permuting Set Valuations

Let us consider V. A function is called a permutation of V if:

(Def. 4) $\mathrm{dom}\,\mathrm{it} = \mathbb{N}$ and for every n holds $\mathrm{it}(n)$ is a permutation of $V(n)$.

In the sequel P is a permutation of V.

Let us consider V, P. The functor $\mathrm{Perm}\,P$ yields a many sorted function from $\mathrm{SetVal}\,V$ into $\mathrm{SetVal}\,V$ and is defined by the conditions (Def. 5).

(Def. 5)(i) $(\mathrm{Perm}\,P)(\mathrm{VERUM}) = \mathrm{id}_1$,

(ii) for every n holds $(\mathrm{Perm}\,P)(\mathrm{prop}\,n) = P(n)$, and

(iii) for all p, q there exists a permutation p' of $\mathrm{SetVal}(V,p)$ and there exists a permutation q' of $\mathrm{SetVal}(V,q)$ such that $p' = (\mathrm{Perm}\,P)(p)$ and $q' = (\mathrm{Perm}\,P)(q)$ and $(\mathrm{Perm}\,P)(p \wedge q) = [:p', q':]$ and $(\mathrm{Perm}\,P)(p \Rightarrow q) = p' \Rightarrow q'$.

Let us consider V, P, p. The functor $\mathrm{Perm}(P,p)$ yields a function from $\mathrm{SetVal}(V,p)$ into $\mathrm{SetVal}(V,p)$ and is defined as follows:

(Def. 6) $\mathrm{Perm}(P,p) = (\mathrm{Perm}\,P)(p)$.

Next we state four propositions:

(33) $\mathrm{Perm}(P,\mathrm{VERUM}) = \mathrm{id}_{\mathrm{SetVal}(V,\mathrm{VERUM})}$.

(34) $\mathrm{Perm}(P,\mathrm{prop}\,n) = P(n)$.

(35) $\mathrm{Perm}(P,p \wedge q) = [\!:\mathrm{Perm}(P,p), \mathrm{Perm}(P,q):\!]$.

(36) For every permutation p' of $\mathrm{SetVal}(V,p)$ and for every permutation q' of $\mathrm{SetVal}(V,q)$ such that $p' = \mathrm{Perm}(P,p)$ and $q' = \mathrm{Perm}(P,q)$ holds $\mathrm{Perm}(P,p \Rightarrow q) = p' \Rightarrow q'$.

Let us consider V, P, p. Observe that $\mathrm{Perm}(P,p)$ is bijective.
One can prove the following propositions:

(37) For every function g from $\mathrm{SetVal}(V,p)$ into $\mathrm{SetVal}(V,q)$ holds $(\mathrm{Perm}(P,p \Rightarrow q))(g) = \mathrm{Perm}(P,q) \cdot g \cdot (\mathrm{Perm}(P,p))^{-1}$.

(38) For every function g from $\mathrm{SetVal}(V,p)$ into $\mathrm{SetVal}(V,q)$ holds $(\mathrm{Perm}(P,p \Rightarrow q))^{-1}(g) = (\mathrm{Perm}(P,q))^{-1} \cdot g \cdot \mathrm{Perm}(P,p)$.

(39) For all functions f, g from $\mathrm{SetVal}(V,p)$ into $\mathrm{SetVal}(V,q)$ such that $f = (\mathrm{Perm}(P,p \Rightarrow q))(g)$ holds $\mathrm{Perm}(P,q) \cdot g = f \cdot \mathrm{Perm}(P,p)$.

(40) Let given V, P be a permutation of V, and x be a set. Suppose x is a fixpoint of $\mathrm{Perm}(P,p)$. Let f be a function. If f is a fixpoint of $\mathrm{Perm}(P,p \Rightarrow q)$, then $f(x)$ is a fixpoint of $\mathrm{Perm}(P,q)$.

6. Canonical Formulae

Let us consider p. We say that p is canonical if and only if:

(Def. 7) For every V there exists a set x such that for every permutation P of V holds x is a fixpoint of $\mathrm{Perm}(P,p)$.

Let us note that VERUM is canonical.
Next we state several propositions:

(41) $p \Rightarrow (q \Rightarrow p)$ is canonical.

(42) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ is canonical.

(43) $p \wedge q \Rightarrow p$ is canonical.

(44) $p \wedge q \Rightarrow q$ is canonical.

(45) $p \Rightarrow (q \Rightarrow p \wedge q)$ is canonical.

(46) If p is canonical and $p \Rightarrow q$ is canonical, then q is canonical.

(47) If $p \in \mathrm{HP\_TAUT}$, then p is canonical.

Let us mention that there exists an element of HP-WFF which is canonical.

7. PSEUDO-CANONICAL FORMULAE

Let us consider p. We say that p is pseudo-canonical if and only if:

(Def. 8) For every V and for every permutation P of V holds there exists a set which is a fixpoint of $\mathrm{Perm}(P, p)$.

Let us observe that every element of HP-WFF which is canonical is also pseudo-canonical.

One can prove the following propositions:

(48) $p \Rightarrow (q \Rightarrow p)$ is pseudo-canonical.

(49) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ is pseudo-canonical.

(50) $p \wedge q \Rightarrow p$ is pseudo-canonical.

(51) $p \wedge q \Rightarrow q$ is pseudo-canonical.

(52) $p \Rightarrow (q \Rightarrow p \wedge q)$ is pseudo-canonical.

(53) If p is pseudo-canonical and $p \Rightarrow q$ is pseudo-canonical, then q is pseudo-canonical.

(54) Let given p, q, given V, and P be a permutation of V. Suppose there exists a set which is a fixpoint of $\mathrm{Perm}(P, p)$ and there exists no set which is a fixpoint of $\mathrm{Perm}(P, q)$. Then $p \Rightarrow q$ is not pseudo-canonical.

(55) $((\mathrm{prop}\,0 \Rightarrow \mathrm{prop}\,1) \Rightarrow \mathrm{prop}\,0) \Rightarrow \mathrm{prop}\,0$ is not pseudo-canonical.

REFERENCES

[1] Grzegorz Bancerek. Curried and uncurried functions. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/funct_5.html`.

[2] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/card_3.html`.

[3] Grzegorz Bancerek. Cartesian product of functions. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/funct_6.html`.

[4] Józef Białas. Group and field definitions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/realset1.html`.

[5] Czesław Byliński. Basic functions and operations on functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_3.html`.

[6] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[7] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[8] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[9] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[10] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/funct_4.html`.

[11] Adam Grabowski. Hilbert positive propositional calculus. *Journal of Formalized Mathematics*, 11, 1999. `http://mizar.org/JFM/Vol11/hilbert1.html`.

[12] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/msualg_3.html`.

[13] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/pralg_1.html`.

[14] Beata Madras. Products of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/pralg_2.html`.

[15] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/pre_circ.html`.

[16] Piotr Rudnicki and Andrzej Trybulec. Fixpoints in complete lattices. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/knaster.html`.

[17] Piotr Rudnicki and Andrzej Trybulec. Abian's fixed point theorem. *Journal of Formalized Mathematics*, 9, 1997. `http://mizar.org/JFM/Vol9/abian.html`.

[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[19] Andrzej Trybulec. Function domains and Frænkel operator. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/fraenkel.html`.

[20] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/pboole.html`.

[21] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/msualg_1.html`.

[22] Andrzej Trybulec. Defining by structural induction in the positive propositional language. *Journal of Formalized Mathematics*, 11, 1999. `http://mizar.org/JFM/Vol11/hilbert2.html`.

[23] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[24] Michał J. Trybulec. Integers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/int_1.html`.

[25] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[26] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

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