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The Incompleteness of the Lattice of Substitutions

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Summary. In [12] we proved that the lattice of substitutions, as defined in [10], is a Heyting lattice (i.e. it is pseudo-complemented and it has the zero element). We show that the lattice needs not to be complete. Obviously, the example has to be infinite, namely we can take the set of natural numbers as variables and a singleton as a set of constants. The incompleteness has been shown for lattices of substitutions defined in terms of [23] and relational structures [20].

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The articles [16], [8], [21], [18], [22], [6], [20], [1], [5], [9], [19], [7], [23], [15], [10], [17], [2], [4], [14], [13], [3], and [11] provide the notation and terminology for this paper.

1. Preliminaries

The scheme *SSubsetUniq* deals with a relational structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> Let A_1, A_2 be subsets of $\mathcal{A}$. Suppose for every set x holds $x \in A_1$ iff $\mathcal{P}[x]$ and for every set x holds $x \in A_2$ iff $\mathcal{P}[x]$. Then $A_1 = A_2$

for all values of the parameters.

Let A, x be sets. Note that $[:A, \{x\}:]$ is function-like.

Next we state several propositions:

(1) For every odd natural number n holds $1 \leq n$.

(2) For every finite non empty subset X of $\mathbb{N}$ holds $\max X \in X$.

(3) For every finite non empty subset X of $\mathbb{N}$ there exists a natural number n such that $X \subseteq \mathrm{Seg}\, n \cup \{0\}$.

(4) For every finite subset X of $\mathbb{N}$ there exists an odd natural number k such that $k \notin X$.

(5) Let k be a natural number and X be a finite non empty subset of $[:\mathbb{N}, \{k\}:]$. Then there exists a non empty natural number n such that $X \subseteq [:\mathrm{Seg}\, n \cup \{0\}, \{k\}:]$.

(6) Let m be a natural number and X be a finite non empty subset of $[:\mathbb{N}, \{m\}:]$. Then there exists a non empty natural number k such that $\langle 2 \cdot k + 1, m \rangle \notin X$.

(7) Let m be a natural number and X be a finite subset of $[:\mathbb{N}, \{m\}:]$. Then there exists a natural number k such that for every natural number l such that $l \geq k$ holds $\langle l, m \rangle \notin X$.

(8) For every upper-bounded lattice L holds $\top_L = \top_{\mathrm{Poset}(L)}$.

(9) For every lower-bounded lattice L holds $\bot_L = \bot_{\mathrm{Poset}(L)}$.

2. Poset of Substitutions

The following four propositions are true:

(11)[1] For every set V and for every finite set C and for all elements A, B of $\mathrm{Fin}(V \dot{\to} C)$ such that $A = \emptyset$ and $B \neq \emptyset$ holds $B \rightarrowtail A = \emptyset$.

(12) For all sets V, V', C, C' such that $V \subseteq V'$ and $C \subseteq C'$ holds $\mathrm{SubstitutionSet}(V,C) \subseteq \mathrm{SubstitutionSet}(V',C')$.

(13) Let V, V', C, C' be sets, A be an element of $\mathrm{Fin}(V \dot{\to} C)$, and B be an element of $\mathrm{Fin}(V' \dot{\to} C')$. If $V \subseteq V'$ and $C \subseteq C'$ and $A = B$, then $\mu A = \mu B$.

(14) Let V, V', C, C' be sets. Suppose $V \subseteq V'$ and $C \subseteq C'$. Then the join operation of $\mathrm{SubstLatt}(V,C) =$ (the join operation of $\mathrm{SubstLatt}(V',C')) \restriction [\!:$ the carrier of $\mathrm{SubstLatt}(V,C)$, the carrier of $\mathrm{SubstLatt}(V,C) :\!]$.

Let V, C be sets. The functor $\mathrm{SubstPoset}(V,C)$ yields a relational structure and is defined by:

(Def. 1) $\mathrm{SubstPoset}(V,C) = \mathrm{Poset}(\mathrm{SubstLatt}(V,C))$.

Let V, C be sets. One can verify that $\mathrm{SubstPoset}(V,C)$ has l.u.b.'s and g.l.b.'s.

Let V, C be sets. One can verify that $\mathrm{SubstPoset}(V,C)$ is reflexive, antisymmetric, and transitive.

We now state two propositions:

(15) Let V, C be sets and a, b be elements of $\mathrm{SubstPoset}(V,C)$. Then $a \leq b$ if and only if for every set x such that $x \in a$ there exists a set y such that $y \in b$ and $y \subseteq x$.

(16) For all sets V, V', C, C' such that $V \subseteq V'$ and $C \subseteq C'$ holds $\mathrm{SubstPoset}(V,C)$ is a full relational substructure of $\mathrm{SubstPoset}(V',C')$.

Let n, k be natural numbers. The functor $\mathrm{PF_A}(n,k)$ yielding an element of $\mathbb{N} \dot{\to} \{k\}$ is defined by:

(Def. 2) For every set x holds $x \in \mathrm{PF_A}(n,k)$ iff there exists an odd natural number m such that $m \leq 2 \cdot n$ and $\langle m, k \rangle = x$ or $\langle 2 \cdot n, k \rangle = x$.

Let n, k be natural numbers. Observe that $\mathrm{PF_A}(n,k)$ is finite.

Let n, k be natural numbers. The functor $\mathrm{PF_C}(n,k)$ yields an element of $\mathbb{N} \dot{\to} \{k\}$ and is defined by:

(Def. 3) For every set x holds $x \in \mathrm{PF_C}(n,k)$ iff there exists an odd natural number m such that $m \leq 2 \cdot n + 1$ and $\langle m, k \rangle = x$.

Let n, k be natural numbers. Note that $\mathrm{PF_C}(n,k)$ is finite.

Next we state four propositions:

(17) For all natural numbers n, k holds $\langle 2 \cdot n + 1, k \rangle \in \mathrm{PF_C}(n,k)$.

(18) For all natural numbers n, k holds $\mathrm{PF_C}(n,k)$ misses $\{\langle 2 \cdot n + 3, k \rangle\}$.

(19) For all natural numbers n, k holds $\mathrm{PF_C}(n+1,k) = \mathrm{PF_C}(n,k) \cup \{\langle 2 \cdot n + 3, k \rangle\}$.

(20) For all natural numbers n, k holds $\mathrm{PF_C}(n,k) \subset \mathrm{PF_C}(n+1,k)$.

Let n, k be natural numbers. Observe that $\mathrm{PF_A}(n,k)$ is non empty.

We now state three propositions:

(21) For all natural numbers n, m, k holds $\mathrm{PF_A}(n,m) \not\subseteq \mathrm{PF_C}(k,m)$.

(22) For all natural numbers n, m, k such that $n \leq k$ holds $\mathrm{PF_C}(n,m) \subseteq \mathrm{PF_C}(k,m)$.

[1] The proposition (10) has been removed.

(23) For every natural number n holds $\mathrm{PF_A}(1,n) = \{\langle 1, n\rangle, \langle 2, n\rangle\}$.

Let n, k be natural numbers. The functor $\mathrm{PF_B}(n,k)$ yields an element of $\mathrm{Fin}(\mathbb{N} \dot{\rightarrow} \{k\})$ and is defined as follows:

(Def. 4) For every set x holds $x \in \mathrm{PF_B}(n,k)$ iff there exists a non empty natural number m such that $m \leq n$ and $x = \mathrm{PF_A}(m,k)$ or $x = \mathrm{PF_C}(n,k)$.

One can prove the following four propositions:

(24) For all natural numbers n, k and for every set x such that $x \in \mathrm{PF_B}(n+1,k)$ there exists a set y such that $y \in \mathrm{PF_B}(n,k)$ and $y \subseteq x$.

(25) For all natural numbers n, k holds $\mathrm{PF_C}(n,k) \notin \mathrm{PF_B}(n+1,k)$.

(26) For all natural numbers n, m, k such that $\mathrm{PF_A}(n,m) \subseteq \mathrm{PF_A}(k,m)$ holds $n = k$.

(27) For all natural numbers n, m, k holds $\mathrm{PF_C}(n,m) \subseteq \mathrm{PF_A}(k,m)$ iff $n < k$.

3. The Incompleteness

One can prove the following proposition

(28) For all natural numbers n, k holds $\mathrm{PF_B}(n,k)$ is an element of $\mathrm{SubstPoset}(\mathbb{N}, \{k\})$.

Let k be a natural number. The functor $\mathrm{PF_D}(k)$ yields a subset of $\mathrm{SubstPoset}(\mathbb{N}, \{k\})$ and is defined as follows:

(Def. 5) For every set x holds $x \in \mathrm{PF_D}(k)$ iff there exists a non empty natural number n such that $x = \mathrm{PF_B}(n,k)$.

The following propositions are true:

(29) For every natural number k holds $\mathrm{PF_B}(1,k) = \{\mathrm{PF_A}(1,k), \mathrm{PF_C}(1,k)\}$.

(30) For every natural number k holds $\mathrm{PF_B}(1,k) \neq \{\emptyset\}$.

Let k be a natural number. One can check that $\mathrm{PF_B}(1,k)$ is non empty.
One can prove the following four propositions:

(31) For all natural numbers n, k holds $\{\mathrm{PF_A}(n,k)\}$ is an element of $\mathrm{SubstPoset}(\mathbb{N}, \{k\})$.

(32) Let k be a natural number, V, X be sets, and a be an element of $\mathrm{SubstPoset}(V, \{k\})$. If $X \in a$, then X is a finite subset of $[: V, \{k\} :]$.

(33) Let m be a natural number and a be an element of $\mathrm{SubstPoset}(\mathbb{N}, \{m\})$. Suppose $\mathrm{PF_D}(m) \geq a$. Let X be a non empty set. If $X \in a$, then it is not true that for every natural number n such that $\langle n, m\rangle \in X$ holds n is odd.

(34) Let k be a natural number, a, b be elements of $\mathrm{SubstPoset}(\mathbb{N}, \{k\})$, and X be a subset of $\mathrm{SubstPoset}(\mathbb{N}, \{k\})$. If $a \leq X$ and $b \leq X$, then $a \sqcup b \leq X$.

Let k be a natural number. Note that there exists an element of $\mathrm{SubstPoset}(\mathbb{N}, \{k\})$ which is non empty.
One can prove the following propositions:

(35) For every natural number n and for every element a of $\mathrm{SubstPoset}(\mathbb{N}, \{n\})$ such that $\emptyset \in a$ holds $a = \{\emptyset\}$.

(36) Let k be a natural number and a be a non empty element of $\mathrm{SubstPoset}(\mathbb{N}, \{k\})$. If $a \neq \{\emptyset\}$, then there exists a finite function f such that $f \in a$ and $f \neq \emptyset$.

(37) Let k be a natural number, a be a non empty element of SubstPoset$(\mathbb{N}, \{k\})$, and a' be an element of Fin$(\mathbb{N} \dot{\rightarrow} \{k\})$. If $a \neq \{\emptyset\}$ and $a = a'$, then Involveda' is a finite non empty subset of $\mathbb{N}$.

(38) Let k be a natural number, a be an element of SubstPoset$(\mathbb{N}, \{k\})$, a' be an element of Fin$(\mathbb{N} \dot{\rightarrow} \{k\})$, and B be a finite non empty subset of $\mathbb{N}$. Suppose $B =$ Involveda' and $a' = a$. Let X be a set. If $X \in a$, then for every natural number l such that $l > \max B + 1$ holds $\langle l, k\rangle \notin X$.

(39) For every natural number k holds $\top_{\mathrm{SubstPoset}(\mathbb{N}, \{k\})} = \{\emptyset\}$.

(40) For every natural number k holds $\bot_{\mathrm{SubstPoset}(\mathbb{N}, \{k\})} = \emptyset$.

(41) For every natural number k and for all elements a, b of SubstPoset$(\mathbb{N}, \{k\})$ such that $a \leq b$ and $a = \{\emptyset\}$ holds $b = \{\emptyset\}$.

(42) For every natural number k and for all elements a, b of SubstPoset$(\mathbb{N}, \{k\})$ such that $a \leq b$ and $b = \emptyset$ holds $a = \emptyset$.

(43) For every natural number m and for every element a of SubstPoset$(\mathbb{N}, \{m\})$ such that $\mathrm{PF}_{\mathrm{D}}(m) \geq a$ holds $a \neq \{\emptyset\}$.

Let m be a natural number. One can verify that SubstPoset$(\mathbb{N}, \{m\})$ is non complete.

Let m be a natural number. Note that SubstLatt$(\mathbb{N}, \{m\})$ is non complete.

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