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Homomorphisms and Isomorphisms of Groups. Quotient Group

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Summary. Quotient group, homomorphisms and isomorphisms of groups are introduced. The so called isomorphism theorems are proved following [9].

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The articles [11], [7], [19], [20], [4], [6], [5], [1], [8], [3], [2], [12], [13], [10], [15], [17], [14], [16], and [18] provide the notation and terminology for this paper.

One can prove the following proposition

(1) Let A, B be non empty sets and f be a function from A into B. Then f is one-to-one if and only if for all elements a, b of A such that $f(a) = f(b)$ holds $a = b$.

Let G be a group and let A be a subgroup of G. We see that the subgroup of A is a subgroup of G.

Let G be a group. One can verify that $\{\mathbf{1}\}_G$ is normal and Ω_G is normal.

For simplicity, we follow the rules: n denotes a natural number, i denotes an integer, G, H, I denote groups, A, B denote subgroups of G, N denotes a normal subgroup of G, a, a_1, a_2, a_3, b denote elements of G, c denotes an element of H, x denotes a set, and A_1, A_2 denote subsets of G.

We now state several propositions:

(2) Let X be a subgroup of A and x be an element of A. Suppose $x = a$. Then $x \cdot X = a \cdot (X$ **qua** subgroup of $G)$ and $X \cdot x = (X$ **qua** subgroup of $G) \cdot a$.

(3) For all subgroups X, Y of A holds $(X$ **qua** subgroup of $G) \cap (Y$ **qua** subgroup of $G) = X \cap Y$.

(4) $a \cdot b \cdot a^{-1} = b^{a^{-1}}$ and $a \cdot (b \cdot a^{-1}) = b^{a^{-1}}$.

(6)[1] $a \cdot A \cdot A = a \cdot A$ and $a \cdot (A \cdot A) = a \cdot A$ and $A \cdot A \cdot a = A \cdot a$ and $A \cdot (A \cdot a) = A \cdot a$.

(7) Let G be a group and A_1 be a subset of G. If $A_1 = \{[a,b] : a$ ranges over elements of G, b ranges over elements of $G\}$, then $G^{\mathrm{c}} = \mathrm{gr}(A_1)$.

(8) Let G be a strict group and B be a strict subgroup of G. Then G^{c} is a subgroup of B if and only if for all elements a, b of G holds $[a,b] \in B$.

(9) For every normal subgroup N of G such that N is a subgroup of B holds N is a normal subgroup of B.

[1] The proposition (5) has been removed.

Let us consider G, B and let M be a normal subgroup of G. Let us assume that the groupoid of M is a subgroup of B. The functor $(M)_B$ yields a strict normal subgroup of B and is defined by:

(Def. 1) $(M)_B =$ the groupoid of M.

We now state the proposition

(10) $B \cap N$ is a normal subgroup of B and $N \cap B$ is a normal subgroup of B.

Let us consider G, B and let N be a normal subgroup of G. Then $B \cap N$ is a strict normal subgroup of B.

Let us consider G, let N be a normal subgroup of G, and let us consider B. Then $N \cap B$ is a strict normal subgroup of B.

Let G be a non empty 1-sorted structure. Let us observe that G is trivial if and only if:

(Def. 2) There exists x such that the carrier of $G = \{x\}$.

Let us observe that there exists a group which is trivial.

Next we state three propositions:

(11) $\{\mathbf{1}\}_G$ is trivial.

(12) G is trivial iff $\mathrm{ord}(G) = 1$ and G is finite.

(13) For every strict group G such that G is trivial holds $\{\mathbf{1}\}_G = G$.

Let us consider G, N. The functor $\mathrm{Cosets}\,N$ yielding a set is defined as follows:

(Def. 3) $\mathrm{Cosets}\,N =$ the left cosets of N.

Let us consider G, N. Observe that $\mathrm{Cosets}\,N$ is non empty.

We now state several propositions:

(14) For every normal subgroup N of G holds $\mathrm{Cosets}\,N =$ the left cosets of N and $\mathrm{Cosets}\,N =$ the right cosets of N.

(15) For every normal subgroup N of G such that $x \in \mathrm{Cosets}\,N$ there exists a such that $x = a \cdot N$ and $x = N \cdot a$.

(16) For every normal subgroup N of G holds $a \cdot N \in \mathrm{Cosets}\,N$ and $N \cdot a \in \mathrm{Cosets}\,N$.

(17) For every normal subgroup N of G such that $x \in \mathrm{Cosets}\,N$ holds x is a subset of G.

(18) For every normal subgroup N of G such that $A_1 \in \mathrm{Cosets}\,N$ and $A_2 \in \mathrm{Cosets}\,N$ holds $A_1 \cdot A_2 \in \mathrm{Cosets}\,N$.

Let us consider G and let N be a normal subgroup of G. The functor $\mathrm{CosOp}\,N$ yields a binary operation on $\mathrm{Cosets}\,N$ and is defined as follows:

(Def. 4) For all elements W_1, W_2 of $\mathrm{Cosets}\,N$ and for all A_1, A_2 such that $W_1 = A_1$ and $W_2 = A_2$ holds $(\mathrm{CosOp}\,N)(W_1, W_2) = A_1 \cdot A_2$.

Let us consider G and let N be a normal subgroup of G. The functor ${}^{G}/_{N}$ yields a groupoid and is defined by:

(Def. 5) ${}^{G}/_{N} = \langle \mathrm{Cosets}\,N, \mathrm{CosOp}\,N \rangle$.

Let us consider G and let N be a normal subgroup of G. Observe that ${}^{G}/_{N}$ is strict and non empty.

Next we state two propositions:

(22)[2] For every normal subgroup N of G holds the carrier of ${}^{G}/_{N} = \mathrm{Cosets}\,N$.

[2] The propositions (19)–(21) have been removed.

(23) For every normal subgroup N of G holds the multiplication of ${}^{G}/_{N} = \mathrm{CosOp}N$.

In the sequel N denotes a normal subgroup of G and S, T_1, T_2 denote elements of ${}^{G}/_{N}$.
Let us consider G, N, S. The functor ${}^{@}S$ yields a subset of G and is defined as follows:

(Def. 6) ${}^{@}S = S$.

The following two propositions are true:

(24) For every normal subgroup N of G and for all elements T_1, T_2 of ${}^{G}/_{N}$ holds $({}^{@}T_1)\cdot({}^{@}T_2) = T_1 \cdot T_2$.

(25) ${}^{@}T_1 \cdot T_2 = ({}^{@}T_1)\cdot({}^{@}T_2)$.

Let us consider G and let N be a normal subgroup of G. One can verify that ${}^{G}/_{N}$ is associative and group-like.
We now state a number of propositions:

(26) For every normal subgroup N of G and for every element S of ${}^{G}/_{N}$ there exists a such that $S = a \cdot N$ and $S = N \cdot a$.

(27) $N \cdot a$ is an element of ${}^{G}/_{N}$ and $a \cdot N$ is an element of ${}^{G}/_{N}$ and $\overline{N}$ is an element of ${}^{G}/_{N}$.

(28) For every normal subgroup N of G holds $x \in {}^{G}/_{N}$ iff there exists a such that $x = a \cdot N$ and $x = N \cdot a$.

(29) For every normal subgroup N of G holds $1_{{}^{G}/_{N}} = \overline{N}$.

(30) For every normal subgroup N of G and for every element S of ${}^{G}/_{N}$ such that $S = a \cdot N$ holds $S^{-1} = a^{-1} \cdot N$.

(31) For every normal subgroup N of G such that the left cosets of N is finite holds ${}^{G}/_{N}$ is finite.

(32) For every normal subgroup N of G holds $\mathrm{Ord}({}^{G}/_{N}) = |\bullet : N|$.

(33) For every normal subgroup N of G such that the left cosets of N is finite holds $\mathrm{ord}({}^{G}/_{N}) = |\bullet : N|_{\mathbb{N}}$.

(34) For every strict normal subgroup M of G such that M is a subgroup of B holds ${}^{B}/_{(M)_B}$ is a subgroup of ${}^{G}/_{M}$.

(35) Let N, M be strict normal subgroups of G. If M is a subgroup of N, then ${}^{N}/_{(M)_N}$ is a normal subgroup of ${}^{G}/_{M}$.

(36) Let G be a strict group and N be a strict normal subgroup of G. Then ${}^{G}/_{N}$ is a commutative group if and only if G^{c} is a subgroup of N.

Let us consider G, H. A function from the carrier of G into the carrier of H is said to be a homomorphism from G to H if:

(Def. 7) $\mathrm{It}(a \cdot b) = \mathrm{it}(a) \cdot \mathrm{it}(b)$.

In the sequel g, h are homomorphisms from G to H, g_1 is a homomorphism from H to G, and h_1 is a homomorphism from H to I.
We now state several propositions:

(40)[3] $g(1_G) = 1_H$.

(41) $g(a^{-1}) = g(a)^{-1}$.

(42) $g(a^b) = g(a)^{g(b)}$.

[3] The propositions (37)–(39) have been removed.

(43) $g([a,b]) = [g(a), g(b)]$.

(44) $g([a_1, a_2, a_3]) = [g(a_1), g(a_2), g(a_3)]$.

(45) $g(a^n) = g(a)^n$.

(46) $g(a^i) = g(a)^i$.

(47) $\mathrm{id}_{\text{the carrier of } G}$ is a homomorphism from G to G.

(48) $h_1 \cdot h$ is a homomorphism from G to I.

Let us consider G, H, I, h, h_1. Then $h_1 \cdot h$ is a homomorphism from G to I.
Let us consider G, H, g. Then $\mathrm{rng}\, g$ is a subset of H.
Let us consider G, H. The functor $G \to \{\mathbf{1}\}_H$ yields a homomorphism from G to H and is defined by:

(Def. 8) For every a holds $(G \to \{\mathbf{1}\}_H)(a) = 1_H$.

Next we state the proposition

(49) $h_1 \cdot (G \to \{\mathbf{1}\}_H) = G \to \{\mathbf{1}\}_I$ and $(H \to \{\mathbf{1}\}_I) \cdot h = G \to \{\mathbf{1}\}_I$.

Let us consider G and let N be a normal subgroup of G. The canonical homomorphism onto cosets of N yields a homomorphism from G to ${}^G/_N$ and is defined by:

(Def. 9) For every a holds (the canonical homomorphism onto cosets of $N)(a) = a \cdot N$.

Let us consider G, H, g. The functor $\mathrm{Ker}\, g$ yields a strict subgroup of G and is defined as follows:

(Def. 10) The carrier of $\mathrm{Ker}\, g = \{a : g(a) = 1_H\}$.

Let us consider G, H, g. Observe that $\mathrm{Ker}\, g$ is normal.
The following propositions are true:

(50) $a \in \mathrm{Ker}\, h$ iff $h(a) = 1_H$.

(51) For all strict groups G, H holds $\mathrm{Ker}(G \to \{\mathbf{1}\}_H) = G$.

(52) For every strict normal subgroup N of G holds Ker(the canonical homomorphism onto cosets of $N) = N$.

Let us consider G, H, g. The functor $\mathrm{Im}\, g$ yields a strict subgroup of H and is defined as follows:

(Def. 11) The carrier of $\mathrm{Im}\, g = g^\circ$(the carrier of G).

Next we state a number of propositions:

(53) $\mathrm{rng}\, g =$ the carrier of $\mathrm{Im}\, g$.

(54) $x \in \mathrm{Im}\, g$ iff there exists a such that $x = g(a)$.

(55) $\mathrm{Im}\, g = \mathrm{gr}(\mathrm{rng}\, g)$.

(56) $\mathrm{Im}(G \to \{\mathbf{1}\}_H) = \{\mathbf{1}\}_H$.

(57) For every normal subgroup N of G holds Im(the canonical homomorphism onto cosets of $N) = {}^G/_N$.

(58) h is a homomorphism from G to $\mathrm{Im}\, h$.

(59) If G is finite, then $\mathrm{Im}\, g$ is finite.

(60) If G is a commutative group, then $\mathrm{Im}\, g$ is commutative.

(61) $\mathrm{Ord}(\mathrm{Im}\, g) \leq \mathrm{Ord}(G)$.

(62) If G is finite, then $\mathrm{ord}(\mathrm{Im}\, g) \leq \mathrm{ord}(G)$.

Let us consider G, H, h. We say that h is monomorphism if and only if:

(Def. 12) h is one-to-one.

We introduce h is a monomorphism as a synonym of h is monomorphism. We say that h is epimorphism if and only if:

(Def. 13) $\mathrm{rng}\, h =$ the carrier of H.

We introduce h is an epimorphism as a synonym of h is epimorphism.
The following propositions are true:

(63) If h is a monomorphism and $c \in \mathrm{Im}\, h$, then $h(h^{-1}(c)) = c$.

(64) If h is a monomorphism, then $h^{-1}(h(a)) = a$.

(65) If h is a monomorphism, then h^{-1} is a homomorphism from $\mathrm{Im}\, h$ to G.

(66) h is a monomorphism iff $\mathrm{Ker}\, h = \{\mathbf{1}\}_G$.

(67) For every strict group H and for every homomorphism h from G to H holds h is an epimorphism iff $\mathrm{Im}\, h = H$.

(68) Let H be a strict group and h be a homomorphism from G to H. Suppose h is an epimorphism. Let c be an element of H. Then there exists a such that $h(a) = c$.

(69) For every normal subgroup N of G holds the canonical homomorphism onto cosets of N is an epimorphism.

Let us consider G, H, h. We say that h is isomorphism if and only if:

(Def. 14) h is an epimorphism and a monomorphism.

We introduce h is an isomorphism as a synonym of h is isomorphism.
The following propositions are true:

(70) h is an isomorphism iff $\mathrm{rng}\, h =$ the carrier of H and h is one-to-one.

(71) If h is an isomorphism, then $\mathrm{dom}\, h =$ the carrier of G and $\mathrm{rng}\, h =$ the carrier of H.

(72) Let H be a strict group and h be a homomorphism from G to H. If h is an isomorphism, then h^{-1} is a homomorphism from H to G.

(73) If h is an isomorphism and $g_1 = h^{-1}$, then g_1 is an isomorphism.

(74) If h is an isomorphism and h_1 is an isomorphism, then $h_1 \cdot h$ is an isomorphism.

(75) For every group G holds the canonical homomorphism onto cosets of $\{\mathbf{1}\}_G$ is an isomorphism.

Let us consider G, H. We say that G and H are isomorphic if and only if:

(Def. 15) There exists h which is an isomorphism.

Let us note that the predicate G and H are isomorphic is reflexive.
One can prove the following propositions:

(77)[4] For all strict groups G, H such that G and H are isomorphic holds H and G are isomorphic.

[4] The proposition (76) has been removed.

(78) If G and H are isomorphic and H and I are isomorphic, then G and I are isomorphic.

(79) If h is a monomorphism, then G and $\mathrm{Im}\, h$ are isomorphic.

(80) For all strict groups G, H such that G is trivial and H is trivial holds G and H are isomorphic.

(81) $\{\mathbf{1}\}_G$ and $\{\mathbf{1}\}_H$ are isomorphic.

(82) For every strict group G holds G and ${}^{G}/_{\{\mathbf{1}\}_G}$ are isomorphic and ${}^{G}/_{\{\mathbf{1}\}_G}$ and G are isomorphic.

(83) For every group G holds ${}^{G}/_{\Omega_G}$ is trivial.

(84) Let G, H be strict groups and h be a homomorphism from G to H. If G and H are isomorphic, then $\mathrm{Ord}(G) = \mathrm{Ord}(H)$.

(85) Let G, H be strict groups. Suppose G and H are isomorphic but G is finite or H is finite. Then G is finite and H is finite.

(86) For all strict groups G, H such that G and H are isomorphic but G is finite or H is finite holds $\mathrm{ord}(G) = \mathrm{ord}(H)$.

(87) For all strict groups G, H such that G and H are isomorphic and G is trivial holds H is trivial.

(88) Let G, H be strict groups. Suppose G and H are isomorphic but G is trivial or H is trivial. Then G is trivial and H is trivial.

(89) Let G, H be strict groups and h be a homomorphism from G to H. Suppose G and H are isomorphic but G is a commutative group or H is a commutative group. Then G is a commutative group and H is a commutative group.

(90) ${}^{G}/_{\mathrm{Ker}\, g}$ and $\mathrm{Im}\, g$ are isomorphic and $\mathrm{Im}\, g$ and ${}^{G}/_{\mathrm{Ker}\, g}$ are isomorphic.

(91) There exists a homomorphism h from ${}^{G}/_{\mathrm{Ker}\, g}$ to $\mathrm{Im}\, g$ such that h is an isomorphism and $g = h \cdot$ the canonical homomorphism onto cosets of $\mathrm{Ker}\, g$.

(92) Let M be a strict normal subgroup of G and J be a strict normal subgroup of ${}^{G}/_{M}$. Suppose $J = {}^{N}/_{(M)_N}$ and M is a subgroup of N. Then ${}^{({}^{G}/_{M})}/_{J}$ and ${}^{G}/_{N}$ are isomorphic.

(93) For every strict normal subgroup N of G holds ${}^{(B \sqcup N)}/_{(N)_{B \sqcup N}}$ and ${}^{B}/_{(B \cap N)}$ are isomorphic.

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