JOURNAL OF FORMALIZED MATHEMATICS
Volume 15, Released 2003, Published 2003
Inst. of Computer Science, Univ. of Białystok

Characterization and Existence of Gröbner Bases

Christoph Schwarzweller
University of Tübingen

Summary. We continue the Mizar formalization of Gröbner bases following [8]. In this article we prove a number of characterizations of Gröbner bases among them that Gröbner bases are convergent rewriting systems. We also show the existence and uniqueness of reduced Gröbner bases.

MML Identifier: GROEB_1.

WWW: http://mizar.org/JFM/Vol15/groeb_1.html

The articles [24], [32], [34], [33], [10], [4], [18], [28], [11], [30], [12], [14], [5], [2], [31], [9], [7], [16], [17], [13], [21], [20], [25], [27], [19], [1], [6], [15], [23], [29], [26], [3], and [22] provide the notation and terminology for this paper.

1. Preliminaries

Let n be an ordinal number, let L be a right zeroed add-associative right complementable unital distributive non trivial double loop structure, and let p be a polynomial of n, L. Then $\{p\}$ is a non empty finite subset of Polynom-Ring(n,L).

We now state several propositions:

(1) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and f, p, g be polynomials of n, L. Suppose f reduces to g, p, T. Then there exists a monomial m of n, L such that $g = f - m * p$.

(2) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and f, p, g be polynomials of n, L. Suppose f reduces to g, p, T. Then there exists a monomial m of n, L such that $g = f - m * p$ and $\mathrm{HT}(m * p, T) \notin \mathrm{Support}\, g$ and $\mathrm{HT}(m * p, T) \leq_T \mathrm{HT}(f, T)$.

(3) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, f, g be polynomials of n, L, and P, Q be subsets of Polynom-Ring(n,L). If $P \subseteq Q$, then if f reduces to g, P, T, then f reduces to g, Q, T.

(4) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and P, Q be subsets of Polynom-Ring(n,L). If $P \subseteq Q$, then PolyRedRel$(P,T) \subseteq$ PolyRedRel(Q,T).

(5) Let n be an ordinal number, L be a right zeroed add-associative right complementable non empty double loop structure, and p be a polynomial of n, L. Then $\mathrm{Support}(-p) = \mathrm{Support}\, p$.

(6) Let n be an ordinal number, T be a connected term order of n, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and p be a polynomial of n, L. Then $\mathrm{HT}(-p,T) = \mathrm{HT}(p,T)$.

(7) Let n be an ordinal number, T be an admissible connected term order of n, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and p, q be polynomials of n, L. Then $\mathrm{HT}(p-q,T) \leq_T \max_T(\mathrm{HT}(p,T),\mathrm{HT}(q,T))$.

(8) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and p, q be polynomials of n, L. If $\mathrm{Support}\, q \subseteq \mathrm{Support}\, p$, then $q \leq_T p$.

(9) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and f, p be non-zero polynomials of n, L. If f is reducible wrt p, T, then $\mathrm{HT}(p,T) \leq_T \mathrm{HT}(f,T)$.

2. Characterization of Gröbner Bases

The following propositions are true:

(10) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure, and p be a polynomial of n, L. Then $\mathrm{PolyRedRel}(\{p\},T)$ is locally-confluent.

(11) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and P be a subset of $\text{Polynom-Ring}(n,L)$. Given a polynomial p of n, L such that $p \in P$ and P–ideal $= \{p\}$–ideal. Then $\mathrm{PolyRedRel}(P,T)$ is locally-confluent.

Let n be an ordinal number, let T be a connected term order of n, let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and let P be a subset of $\text{Polynom-Ring}(n,L)$. The functor $\mathrm{HT}(P,T)$ yielding a subset of $\mathrm{Bags}\, n$ is defined as follows:

(Def. 1) $\mathrm{HT}(P,T) = \{\mathrm{HT}(p,T); p$ ranges over polynomials of n, L: $p \in P \wedge p \neq 0_n L\}$.

Let n be an ordinal number and let S be a subset of $\mathrm{Bags}\, n$. The functor $\mathrm{multiples}(S)$ yields a subset of $\mathrm{Bags}\, n$ and is defined as follows:

(Def. 2) $\mathrm{multiples}(S) = \{b; b$ ranges over elements of $\mathrm{Bags}\, n : \bigvee_{b' : \text{bag of } n} (b' \in S \wedge b' \mid b)\}$.

The following propositions are true:

(12) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and P be a subset of $\text{Polynom-Ring}(n,L)$. If $\mathrm{PolyRedRel}(P,T)$ is locally-confluent, then $\mathrm{PolyRedRel}(P,T)$ is confluent.

(13) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and P be a subset of $\text{Polynom-Ring}(n,L)$. If $\mathrm{PolyRedRel}(P,T)$ is confluent, then $\mathrm{PolyRedRel}(P,T)$ has unique normal form property.

(14) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and P be a subset of Polynom-Ring(n,L). Suppose PolyRedRel(P,T) has unique normal form property. Then PolyRedRel(P,T) has Church-Rosser property.

(15) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and P be a non empty subset of Polynom-Ring(n,L). Suppose PolyRedRel(P,T) has Church-Rosser property. Let f be a polynomial of n, L. If $f \in P$–ideal, then PolyRedRel(P,T) reduces f to $0_n L$.

(16) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and P be a subset of Polynom-Ring(n,L). Suppose that for every polynomial f of n, L such that $f \in P$–ideal holds PolyRedRel(P,T) reduces f to $0_n L$. Let f be a non-zero polynomial of n, L. If $f \in P$–ideal, then f is reducible wrt P, T.

(17) Let n be a natural number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and P be a subset of Polynom-Ring(n,L). Suppose that for every non-zero polynomial f of n, L such that $f \in P$–ideal holds f is reducible wrt P, T. Let f be a non-zero polynomial of n, L. If $f \in P$–ideal, then f is top reducible wrt P, T.

(18) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and P be a subset of Polynom-Ring(n,L). Suppose that for every non-zero polynomial f of n, L such that $f \in P$–ideal holds f is top reducible wrt P, T. Let b be a bag of n. If $b \in \mathrm{HT}(P\text{–ideal},T)$, then there exists a bag b' of n such that $b' \in \mathrm{HT}(P,T)$ and $b' \mid b$.

(19) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and P be a subset of Polynom-Ring(n,L). Suppose that for every bag b of n such that $b \in \mathrm{HT}(P\text{–ideal},T)$ there exists a bag b' of n such that $b' \in \mathrm{HT}(P,T)$ and $b' \mid b$. Then $\mathrm{HT}(P\text{–ideal},T) \subseteq \mathrm{multiples}(\mathrm{HT}(P,T))$.

(20) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and P be a subset of Polynom-Ring(n,L). If $\mathrm{HT}(P\text{–ideal},T) \subseteq \mathrm{multiples}(\mathrm{HT}(P,T))$, then PolyRedRel$(P,T)$ is locally-confluent.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let G be a subset of Polynom-Ring(n,L). We say that G is a Groebner basis wrt T if and only if:

(Def. 3) PolyRedRel(G,T) is locally-confluent.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let G, I be subsets of Polynom-Ring(n,L). We say that G is a Groebner basis of I, T if and only if:

(Def. 4) G–ideal $= I$ and PolyRedRel(G,T) is locally-confluent.

The following propositions are true:

(21) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and G, P be non empty subsets of Polynom-Ring(n,L). If G is a Groebner basis of P, T, then PolyRedRel(G,T) is a completion of PolyRedRel(P,T).

(22) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, p, q be elements of Polynom-Ring(n,L), and G be a non empty subset of Polynom-Ring(n,L). Suppose G is a Groebner basis wrt T. Then $p \equiv q (\mathrm{mod}\, G\text{–ideal})$ if and only if $\mathrm{nf}_{\mathrm{PolyRedRel}(G,T)}(p) = \mathrm{nf}_{\mathrm{PolyRedRel}(G,T)}(q)$.

(23) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, f be a polynomial of n, L, and P be a non empty subset of Polynom-Ring(n,L). Suppose P is a Groebner basis wrt T. Then $f \in P$–ideal if and only if PolyRedRel(P,T) reduces f to $0_n L$.

(24) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, I be a subset of Polynom-Ring(n,L), and G be a non empty subset of Polynom-Ring(n,L). Suppose G is a Groebner basis of I, T. Let f be a polynomial of n, L. If $f \in I$, then PolyRedRel(G,T) reduces f to $0_n L$.

(25) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and G, I be subsets of Polynom-Ring(n,L). Suppose that for every polynomial f of n, L such that $f \in I$ holds PolyRedRel(G,T) reduces f to $0_n L$. Let f be a non-zero polynomial of n, L. If $f \in I$, then f is reducible wrt G, T.

(26) Let n be a natural number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, I be an add closed left ideal subset of Polynom-Ring(n,L), and G be a subset of Polynom-Ring(n,L). Suppose $G \subseteq I$. Suppose that for every non-zero polynomial f of n, L such that $f \in I$ holds f is reducible wrt G, T. Let f be a non-zero polynomial of n, L. If $f \in I$, then f is top reducible wrt G, T.

(27) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and G, I be subsets of Polynom-Ring(n,L). Suppose that for every non-zero polynomial f of n, L such that $f \in I$ holds f is top reducible wrt G, T. Let b be a bag of n. If $b \in \mathrm{HT}(I,T)$, then there exists a bag b' of n such that $b' \in \mathrm{HT}(G,T)$ and $b' \mid b$.

(28) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and G, I be subsets of Polynom-Ring(n,L). Suppose that for every bag b of n such that $b \in \mathrm{HT}(I,T)$ there exists a bag b' of n such that $b' \in \mathrm{HT}(G,T)$ and $b' \mid b$. Then $\mathrm{HT}(I,T) \subseteq$ multiples$(\mathrm{HT}(G,T))$.

(29) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, I be an add closed left ideal non empty subset of Polynom-Ring(n,L), and G be a non empty subset of Polynom-Ring(n,L). If $G \subseteq I$, then if $\mathrm{HT}(I,T) \subseteq$ multiples$(\mathrm{HT}(G,T))$, then G is a Groebner basis of I, T.

3. Existence of Gröbner Bases

The following four propositions are true:

(30) Let n be a natural number, T be a connected admissible term order of n, and L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure. Then $\{0_n L\}$ is a Groebner basis of $\{0_n L\}$, T.

(31) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure, and p be a polynomial of n, L. Then $\{p\}$ is a Groebner basis of $\{p\}$–ideal, T.

(32) Let T be an admissible connected term order of $\emptyset$, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, I be an add closed left ideal non empty subset of Polynom-Ring$(\emptyset, L)$, and P be a non empty subset of Polynom-Ring$(\emptyset, L)$. If $P \subseteq I$ and $P \neq \{0_\emptyset L\}$, then P is a Groebner basis of I, T.

(33) Let n be a non empty ordinal number, T be an admissible connected term order of n, and L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure. Then there exists a subset P of Polynom-Ring(n, L) such that P is not a Groebner basis wrt T.

Let n be an ordinal number. The functor DivOrder(n) yields an order in Bags n and is defined by:

(Def. 5) For all bags b_1, b_2 of n holds $\langle b_1, b_2 \rangle \in$ DivOrder(n) iff $b_1 \mid b_2$.

Let n be a natural number. Observe that $\langle \text{Bags}\, n, \text{DivOrder}(n) \rangle$ is Dickson.
The following three propositions are true:

(34) For every natural number n and for every subset N of $\langle \text{Bags}\, n, \text{DivOrder}(n) \rangle$ holds there exists a finite subset of Bags n which is Dickson basis of N, $\langle \text{Bags}\, n, \text{DivOrder}(n) \rangle$.

(35) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and I be an add closed left ideal non empty subset of Polynom-Ring(n, L). Then there exists a finite subset of Polynom-Ring(n, L) which is a Groebner basis of I, T.

(36) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and I be an add closed left ideal non empty subset of Polynom-Ring(n, L). Suppose $I \neq \{0_n L\}$. Then there exists a finite subset G of Polynom-Ring(n, L) such that G is a Groebner basis of I, T and $0_n L \notin G$.

Let n be an ordinal number, let T be a connected term order of n, let L be a non empty multiplicative loop with zero structure, and let p be a polynomial of n, L. We say that p is monic wrt T if and only if:

(Def. 6) $\text{HC}(p, T) = \mathbf{1}_L$.

Let n be an ordinal number, let T be a connected term order of n, let L be a right zeroed add-associative right complementable commutative associative well unital distributive field-like non trivial non empty double loop structure, and let P be a subset of Polynom-Ring(n, L). We say that P is reduced wrt T if and only if:

(Def. 7) For every polynomial p of n, L such that $p \in P$ holds p is monic wrt T and irreducible wrt $P \setminus \{p\}$, T.

We introduce P is autoreduced wrt T as a synonym of P is reduced wrt T.

One can prove the following propositions:

(37) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, I be an add closed left ideal subset of Polynom-Ring(n,L), m be a monomial of n, L, and f, g be polynomials of n, L. Suppose $f \in I$ and $g \in I$ and $\mathrm{HM}(f,T) = m$ and $\mathrm{HM}(g,T) = m$. Suppose that

(i) it is not true that there exists a polynomial p of n, L such that $p \in I$ and $p <_T f$ and $\mathrm{HM}(p,T) = m$, and

(ii) it is not true that there exists a polynomial p of n, L such that $p \in I$ and $p <_T g$ and $\mathrm{HM}(p,T) = m$.

Then $f = g$.

(38) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, I be an add closed left ideal non empty subset of Polynom-Ring(n,L), G be a subset of Polynom-Ring(n,L), p be a polynomial of n, L, and q be a non-zero polynomial of n, L. Suppose $p \in G$ and $q \in G$ and $p \neq q$ and $\mathrm{HT}(q,T) \mid \mathrm{HT}(p,T)$. If G is a Groebner basis of I, T, then $G \setminus \{p\}$ is a Groebner basis of I, T.

(39) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and I be an add closed left ideal non empty subset of Polynom-Ring(n,L). If $I \neq \{0_n L\}$, then there exists a finite subset G of Polynom-Ring(n,L) which is a Groebner basis of I, T and reduced wrt T.

(40) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, I be an add closed left ideal non empty subset of Polynom-Ring(n,L), and G_1, G_2 be non empty finite subsets of Polynom-Ring(n,L). Suppose G_1 is a Groebner basis of I, T and reduced wrt T and G_2 is a Groebner basis of I, T and reduced wrt T. Then $G_1 = G_2$.

REFERENCES

[1] Jonathan Backer, Piotr Rudnicki, and Christoph Schwarzweller. Ring ideals. *Journal of Formalized Mathematics*, 12, 2000. `http://mizar.org/JFM/Vol12/ideal_1.html`.

[2] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[3] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[4] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[5] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/card_3.html`.

[6] Grzegorz Bancerek. Reduction relations. *Journal of Formalized Mathematics*, 7, 1995. `http://mizar.org/JFM/Vol7/rewrite1.html`.

[7] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[8] Thomas Becker and Volker Weispfenning. *Gröbner Bases: A Computational Approach to Commutative Algebra*. Springer-Verlag, New York, Berlin, 1993.

[9] Józef Białas. Group and field definitions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/realset1.html`.

[10] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[11] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[12] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[13] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/vectsp_1.html.

[14] Gilbert Lee and Piotr Rudnicki. Dickson's lemma. *Journal of Formalized Mathematics*, 14, 2002. http://mizar.org/JFM/Vol14/dickson.html.

[15] Gilbert Lee and Piotr Rudnicki. On ordering of bags. *Journal of Formalized Mathematics*, 14, 2002. http://mizar.org/JFM/Vol14/bagorder.html.

[16] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pralg_1.html.

[17] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_2.html.

[18] Michał Muzalewski and Wojciech Skaba. From loops to abelian multiplicative groups with zero. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/algstr_1.html.

[19] Piotr Rudnicki and Andrzej Trybulec. Multivariate polynomials with arbitrary number of variables. *Journal of Formalized Mathematics*, 11, 1999. http://mizar.org/JFM/Vol11/polynom1.html.

[20] Christoph Schwarzweller. The binomial theorem for algebraic structures. *Journal of Formalized Mathematics*, 12, 2000. http://mizar.org/JFM/Vol12/binom.html.

[21] Christoph Schwarzweller. More on multivariate polynomials: Monomials and constant polynomials. *Journal of Formalized Mathematics*, 13, 2001. http://mizar.org/JFM/Vol13/polynom7.html.

[22] Christoph Schwarzweller. Polynomial reduction. *Journal of Formalized Mathematics*, 14, 2002. http://mizar.org/JFM/Vol14/polyred.html.

[23] Christoph Schwarzweller. Term orders. *Journal of Formalized Mathematics*, 14, 2002. http://mizar.org/JFM/Vol14/termord.html.

[24] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[25] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pboole.html.

[26] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[27] Wojciech A. Trybulec. Partially ordered sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/orders_1.html.

[28] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[29] Wojciech A. Trybulec. Groups. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/group_1.html.

[30] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[31] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[32] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[33] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

[34] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_2.html.

Received June 11, 2003

Published January 2, 2004