JOURNAL OF FORMALIZED MATHEMATICS
Volume 15, Released 2003, Published 2003
Inst. of Computer Science, Univ. of Białystok

# The Underlying Principle of Dijkstra's Shortest Path Algorithm

Jing-Chao Chen
Donghua University
Shanghai

Yatsuka Nakamura
Shinshu University
Nagano

**Summary.** A path from a source vertex $v$ to a target vertex $u$ is said to be a shortest path if its total cost is minimum among all $v$-to-$u$ paths. Dijkstra's algorithm is a classic shortest path algorithm, which is described in many textbooks. To justify its correctness (whose rigorous proof will be given in the next article), it is necessary to clarify its underlying principle. For this purpose, the article justifies the following basic facts, which are the core of Dijkstra's algorithm.

- A graph is given, its vertex set is denoted by $V$. Assume $U$ is the subset of $V$, and if a path $p$ from $s$ to $t$ is the shortest among the set of paths, each of which passes through only the vertices in $U$, except the source and sink, and its source and sink is $s$ and in $V$, respectively, then $p$ is a shortest path from $s$ to $t$ in the graph, and for any subgraph which contains at least $U$, it is also the shortest.
- Let $p(s,x,U)$ denote the shortest path from $s$ to $x$ in a subgraph whose the vertex set is the union of $\{s,x\}$ and $U$, and cost $(p)$ denote the cost of path $p(s,x,U)$, $\mathrm{cost}(x,y)$ the cost of the edge from $x$ to $y$. Give $p(s,x,U)$, $q(s,y,U)$ and $r(s,y,U\cup\{x\})$. If $\mathrm{cost}(p) = \min\{\mathrm{cost}(w) : w(s,t,U) \wedge t \in V\}$, then we have

$$\mathrm{cost}(r) = \min(\mathrm{cost}(p)+\mathrm{cost}(x,y), \mathrm{cost}(q)).$$

This is the well-known triangle comparison of Dijkstra's algorithm.

MML Identifier: `GRAPH_5`.

WWW: `http://mizar.org/JFM/Vol15/graph_5.html`

The articles [13], [16], [14], [17], [4], [3], [6], [15], [1], [8], [9], [2], [10], [5], [12], [7], and [11] provide the notation and terminology for this paper.

## 1. Preliminaries

We use the following convention: $n$, $m$, $i$, $j$, $k$ denote natural numbers, $x$, $y$, $e$, $X$, $V$, $U$ denote sets, and $W$, $f$, $g$ denote functions.

The following two propositions are true:

(1) For every finite function $f$ holds $\mathrm{card}\,\mathrm{rng}\, f \leq \mathrm{card}\,\mathrm{dom}\, f$.

(2) If $\mathrm{rng}\, f \subseteq \mathrm{rng}\, g$ and $x \in \mathrm{dom}\, f$, then there exists $y$ such that $y \in \mathrm{dom}\, g$ and $f(x) = g(y)$.

The scheme *LambdaAB* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function $f$ such that $\mathrm{dom}\, f = \mathcal{A}$ and for every element $x$ of $\mathcal{B}$ such that $x \in \mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

for all values of the parameters.

One can prove the following propositions:

(3) Let $D$ be a finite set, $n$ be a natural number, and $X$ be a set. If $X = \{x; x$ ranges over elements of $D^*$: $1 \leq \operatorname{len} x \wedge \operatorname{len} x \leq n\}$, then $X$ is finite.

(4) Let $D$ be a finite set, $n$ be a natural number, and $X$ be a set. If $X = \{x; x$ ranges over elements of $D^*$: $\operatorname{len} x \leq n\}$, then $X$ is finite.

(5) For every finite set $D$ holds $\operatorname{card} D \neq 0$ iff $D \neq \emptyset$.

(6) Let $D$ be a finite set and $k$ be a natural number. Suppose $\operatorname{card} D = k+1$. Then there exists an element $x$ of $D$ and there exists a subset $C$ of $D$ such that $D = C \cup \{x\}$ and $\operatorname{card} C = k$.

(7) For every finite set $D$ such that $\operatorname{card} D = 1$ there exists an element $x$ of $D$ such that $D = \{x\}$.

The scheme *MinValue* deals with a non empty finite set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a real number, and states that:

There exists an element $x$ of $\mathcal{A}$ such that for every element $y$ of $\mathcal{A}$ holds $\mathcal{F}(x) \leq \mathcal{F}(y)$

for all values of the parameters.

Let $D$ be a set and let $X$ be a non empty subset of $D^*$. We see that the element of $X$ is a finite sequence of elements of $D$.

## 2. Additional Properties of Finite Sequences

In the sequel $p$, $q$ are finite sequences.

The following three propositions are true:

(8) $p \neq \emptyset$ iff $\operatorname{len} p \geq 1$.

(9) For all $n$, $m$ such that $1 \leq n$ and $n < m$ and $m \leq \operatorname{len} p$ holds $p(n) \neq p(m)$ iff $p$ is one-to-one.

(10) For all $n$, $m$ such that $1 \leq n$ and $n < m$ and $m \leq \operatorname{len} p$ holds $p(n) \neq p(m)$ iff $\operatorname{card} \operatorname{rng} p = \operatorname{len} p$.

In the sequel $G$ denotes a graph and $p_1$, $q_1$ denote finite sequences of elements of the edges of $G$.

The following two propositions are true:

(11) If $i \in \operatorname{dom} p_1$, then (the source of $G)(p_1(i)) \in$ the vertices of $G$ and (the target of $G)(p_1(i)) \in$ the vertices of $G$.

(12) If $q \frown \langle x \rangle$ is one-to-one and $\operatorname{rng}(q \frown \langle x \rangle) \subseteq \operatorname{rng} p$, then there exist finite sequences $p_2$, $p_3$ such that $p = p_2 \frown \langle x \rangle \frown p_3$ and $\operatorname{rng} q \subseteq \operatorname{rng}(p_2 \frown p_3)$.

## 3. Additional Properties of Chains and Oriented Paths

One can prove the following three propositions:

(13) If $p \frown q$ is a chain of $G$, then $p$ is a chain of $G$ and $q$ is a chain of $G$.

(14) If $p \frown q$ is an oriented chain of $G$, then $p$ is an oriented chain of $G$ and $q$ is an oriented chain of $G$.

(15) Let $p$, $q$ be oriented chains of $G$. Suppose (the target of $G)(p(\operatorname{len} p)) =$ (the source of $G)(q(1))$. Then $p \frown q$ is an oriented chain of $G$.

## 4. Additional Properties of Acyclic Oriented Paths

Next we state several propositions:

(16) $\emptyset$ is a Simple oriented chain of $G$.

(17) Suppose $p \frown q$ is a Simple oriented chain of $G$. Then $p$ is a Simple oriented chain of $G$ and $q$ is a Simple oriented chain of $G$.

(18) If $\operatorname{len} p_1 = 1$, then $p_1$ is a Simple oriented chain of $G$.

(19) Let $p$ be a Simple oriented chain of $G$ and $q$ be a finite sequence of elements of the edges of $G$. Suppose that

(i) $\operatorname{len} p \geq 1$,

(ii) $\operatorname{len} q = 1$,

(iii) (the source of $G)(q(1)) =$ (the target of $G)(p(\operatorname{len} p))$,

(iv) (the source of $G)(p(1)) \neq$ (the target of $G)(p(\operatorname{len} p))$, and

(v) it is not true that there exists $k$ such that $1 \leq k$ and $k \leq \operatorname{len} p$ and (the target of $G)(p(k)) =$ (the target of $G)(q(1))$.

Then $p \frown q$ is a Simple oriented chain of $G$.

(20) Every Simple oriented chain of $G$ is one-to-one.

## 5. The Set of the Vertices On a Path or an Edge

Let $G$ be a graph and let $e$ be an element of the edges of $G$. The functor vertices $e$ is defined by:

(Def. 1) vertices $e = \{$(the source of $G)(e)$, (the target of $G)(e)\}$.

Let us consider $G$, $p_1$. The functor vertices $p_1$ yielding a subset of the vertices of $G$ is defined by:

(Def. 2) vertices $p_1 = \{v; v$ ranges over vertices of $G$: $\bigvee_i$ $(i \in \operatorname{dom} p_1 \ \wedge \ v \in \operatorname{vertices}((p_1)_i))\}$.

One can prove the following propositions:

(21) Let $p$ be a Simple oriented chain of $G$. Suppose $p = p_1 \frown q_1$ and $\operatorname{len} p_1 \geq 1$ and $\operatorname{len} q_1 \geq 1$ and (the source of $G)(p(1)) \neq$ (the target of $G)(p(\operatorname{len} p))$. Then (the source of $G)(p(1)) \notin$ vertices $q_1$ and (the target of $G)(p(\operatorname{len} p)) \notin$ vertices $p_1$.

(22) vertices $p_1 \subseteq V$ iff for every $i$ such that $i \in \operatorname{dom} p_1$ holds $\operatorname{vertices}((p_1)_i) \subseteq V$.

(23) Suppose vertices $p_1 \not\subseteq V$. Then there exists a natural number $i$ and there exist finite sequences $q$, $r$ of elements of the edges of $G$ such that $i + 1 \leq \operatorname{len} p_1$ and $\operatorname{vertices}((p_1)_{i+1}) \not\subseteq V$ and $\operatorname{len} q = i$ and $p_1 = q \frown r$ and vertices $q \subseteq V$.

(24) If $\operatorname{rng} q_1 \subseteq \operatorname{rng} p_1$, then vertices $q_1 \subseteq$ vertices $p_1$.

(25) If $\operatorname{rng} q_1 \subseteq \operatorname{rng} p_1$ and vertices $p_1 \setminus X \subseteq V$, then vertices $q_1 \setminus X \subseteq V$.

(26) If $\operatorname{vertices}(p_1 \frown q_1) \setminus X \subseteq V$, then vertices $p_1 \setminus X \subseteq V$ and vertices $q_1 \setminus X \subseteq V$.

In the sequel $v$, $v_1$, $v_2$, $v_3$ are elements of the vertices of $G$.
One can prove the following propositions:

(27) For every element $e$ of the edges of $G$ such that $v =$ (the source of $G)(e)$ or $v =$ (the target of $G)(e)$ holds $v \in$ vertices $e$.

(28) If $i \in \operatorname{dom} p_1$ and if $v =$ (the source of $G)(p_1(i))$ or $v =$ (the target of $G)(p_1(i))$, then $v \in$ vertices $p_1$.

(29) If $\text{len}\, p_1 = 1$, then $\text{vertices}\, p_1 = \text{vertices}((p_1)_1)$.

(30) $\text{vertices}\, p_1 \subseteq \text{vertices}(p_1 \frown q_1)$ and $\text{vertices}\, q_1 \subseteq \text{vertices}(p_1 \frown q_1)$.

In the sequel $p$, $q$ are oriented chains of $G$.
The following propositions are true:

(31) If $p = q \frown p_1$ and $\text{len}\, q \geq 1$ and $\text{len}\, p_1 = 1$, then $\text{vertices}\, p = \text{vertices}\, q \cup \{(\text{the target of } G)(p_1(1))\}$.

(32) If $v \neq (\text{the source of } G)(p(1))$ and $v \in \text{vertices}\, p$, then there exists $i$ such that $1 \leq i$ and $i \leq \text{len}\, p$ and $v = (\text{the target of } G)(p(i))$.

## 6. Directed Paths between Two Vertices

Let us consider $G$, $p$, $v_1$, $v_2$. We say that $p$ is oriented path from $v_1$ to $v_2$ if and only if:

(Def. 3) $p \neq \emptyset$ and $(\text{the source of } G)(p(1)) = v_1$ and $(\text{the target of } G)(p(\text{len}\, p)) = v_2$.

Let us consider $G$, $v_1$, $v_2$, $p$, $V$. We say that $p$ is oriented path from $v_1$ to $v_2$ in $V$ if and only if:

(Def. 4) $p$ is oriented path from $v_1$ to $v_2$ and $\text{vertices}\, p \setminus \{v_2\} \subseteq V$.

Let $G$ be a graph and let $v_1$, $v_2$ be elements of the vertices of $G$. The functor $\text{OrientedPaths}(v_1, v_2)$ yields a subset of $(\text{the edges of } G)^*$ and is defined as follows:

(Def. 5) $\text{OrientedPaths}(v_1, v_2) = \{p; p$ ranges over oriented chains of $G$: $p$ is oriented path from $v_1$ to $v_2\}$.

We now state several propositions:

(33) If $p$ is oriented path from $v_1$ to $v_2$, then $v_1 \in \text{vertices}\, p$ and $v_2 \in \text{vertices}\, p$.

(34) $x \in \text{OrientedPaths}(v_1, v_2)$ iff there exists $p$ such that $p = x$ and $p$ is oriented path from $v_1$ to $v_2$.

(35) If $p$ is oriented path from $v_1$ to $v_2$ in $V$ and $v_1 \neq v_2$, then $v_1 \in V$.

(36) If $p$ is oriented path from $v_1$ to $v_2$ in $V$ and $V \subseteq U$, then $p$ is oriented path from $v_1$ to $v_2$ in $U$.

(37) Suppose $\text{len}\, p \geq 1$ and $p$ is oriented path from $v_1$ to $v_2$ and $p_1(1)$ orientedly joins $v_2$, $v_3$ and $\text{len}\, p_1 = 1$. Then there exists $q$ such that $q = p \frown p_1$ and $q$ is oriented path from $v_1$ to $v_3$.

(38) Suppose $q = p \frown p_1$ and $\text{len}\, p \geq 1$ and $\text{len}\, p_1 = 1$ and $p$ is oriented path from $v_1$ to $v_2$ in $V$ and $p_1(1)$ orientedly joins $v_2$, $v_3$. Then $q$ is oriented path from $v_1$ to $v_3$ in $V \cup \{v_2\}$.

## 7. Acyclic (or Simple) Paths

Let $G$ be a graph, let $p$ be an oriented chain of $G$, and let $v_1$, $v_2$ be elements of the vertices of $G$. We say that $p$ is acyclic path from $v_1$ to $v_2$ if and only if:

(Def. 6) $p$ is Simple and oriented path from $v_1$ to $v_2$.

Let $G$ be a graph, let $p$ be an oriented chain of $G$, let $v_1$, $v_2$ be elements of the vertices of $G$, and let $V$ be a set. We say that $p$ is acyclic path from $v_1$ to $v_2$ in $V$ if and only if:

(Def. 7) $p$ is Simple and oriented path from $v_1$ to $v_2$ in $V$.

Let $G$ be a graph and let $v_1$, $v_2$ be elements of the vertices of $G$. The functor $\text{AcyclicPaths}(v_1, v_2)$ yields a subset of $(\text{the edges of } G)^*$ and is defined by:

(Def. 8) AcyclicPaths$(v_1, v_2) = \{p; p$ ranges over Simple oriented chains of $G$: $p$ is acyclic path from $v_1$ to $v_2\}$.

Let $G$ be a graph, let $v_1$, $v_2$ be elements of the vertices of $G$, and let $V$ be a set. The functor AcyclicPaths$(v_1, v_2, V)$ yields a subset of (the edges of $G)^*$ and is defined as follows:

(Def. 9) AcyclicPaths$(v_1, v_2, V) = \{p; p$ ranges over Simple oriented chains of $G$: $p$ is acyclic path from $v_1$ to $v_2$ in $V\}$.

Let $G$ be a graph and let $p$ be an oriented chain of $G$. The functor AcyclicPaths$(p)$ yielding a subset of (the edges of $G)^*$ is defined by the condition (Def. 10).

(Def. 10) AcyclicPaths$(p) = \{q; q$ ranges over Simple oriented chains of $G$: $q \neq \emptyset \wedge$ (the source of $G)(q(1)) =$ (the source of $G)(p(1)) \wedge$ (the target of $G)(q(\operatorname{len} q)) =$ (the target of $G)(p(\operatorname{len} p)) \wedge \operatorname{rng} q \subseteq \operatorname{rng} p\}$.

Let $G$ be a graph. The functor AcyclicPaths$(G)$ yields a subset of (the edges of $G)^*$ and is defined as follows:

(Def. 11) AcyclicPaths$(G) = \{q : q$ ranges over Simple oriented chains of $G\}$.

We now state a number of propositions:

(39) If $p = \emptyset$, then $p$ is not acyclic path from $v_1$ to $v_2$.

(40) If $p$ is acyclic path from $v_1$ to $v_2$, then $p$ is oriented path from $v_1$ to $v_2$.

(41) AcyclicPaths$(v_1, v_2) \subseteq$ OrientedPaths$(v_1, v_2)$.

(42) AcyclicPaths$(p) \subseteq$ AcyclicPaths$(G)$.

(43) AcyclicPaths$(v_1, v_2) \subseteq$ AcyclicPaths$(G)$.

(44) If $p$ is oriented path from $v_1$ to $v_2$, then AcyclicPaths$(p) \subseteq$ AcyclicPaths$(v_1, v_2)$.

(45) If $p$ is oriented path from $v_1$ to $v_2$ in $V$, then AcyclicPaths$(p) \subseteq$ AcyclicPaths$(v_1, v_2, V)$.

(46) If $q \in$ AcyclicPaths$(p)$, then $\operatorname{len} q \leq \operatorname{len} p$.

(47) If $p$ is oriented path from $v_1$ to $v_2$, then AcyclicPaths$(p) \neq \emptyset$ and AcyclicPaths$(v_1, v_2) \neq \emptyset$.

(48) If $p$ is oriented path from $v_1$ to $v_2$ in $V$, then AcyclicPaths$(p) \neq \emptyset$ and AcyclicPaths$(v_1, v_2, V) \neq \emptyset$.

(49) AcyclicPaths$(v_1, v_2, V) \subseteq$ AcyclicPaths$(G)$.

## 8. Weight Graphs and Their Basic Properties

The subset $\mathbb{R}_{\geq 0}$ of $\mathbb{R}$ is defined by:

(Def. 12) $\mathbb{R}_{\geq 0} = \{r; r$ ranges over real numbers: $r \geq 0\}$.

Let us observe that $\mathbb{R}_{\geq 0}$ is non empty.

Let $G$ be a graph and let $W$ be a function. We say that $W$ is nonnegative weight of $G$ if and only if:

(Def. 13) $W$ is a function from the edges of $G$ into $\mathbb{R}_{\geq 0}$.

Let $G$ be a graph and let $W$ be a function. We say that $W$ is weight of $G$ if and only if:

(Def. 14) $W$ is a function from the edges of $G$ into $\mathbb{R}$.

Let $G$ be a graph, let $p$ be a finite sequence of elements of the edges of $G$, and let $W$ be a function. Let us assume that $W$ is weight of $G$. The functor RealSequence$(p, W)$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def. 15) $\operatorname{dom} p = \operatorname{dom} \text{RealSequence}(p, W)$ and for every natural number $i$ such that $i \in \operatorname{dom} p$ holds $(\text{RealSequence}(p, W))(i) = W(p(i))$.

Let $G$ be a graph, let $p$ be a finite sequence of elements of the edges of $G$, and let $W$ be a function. The functor $\text{cost}(p, W)$ yielding a real number is defined by:

(Def. 16) $\text{cost}(p, W) = \sum \text{RealSequence}(p, W)$.

We now state a number of propositions:

(50) If $W$ is nonnegative weight of $G$, then $W$ is weight of $G$.

(51) Let $f$ be a finite sequence of elements of $\mathbb{R}$. Suppose $W$ is nonnegative weight of $G$ and $f = \text{RealSequence}(p_1, W)$. Let given $i$. If $i \in \operatorname{dom} f$, then $f(i) \geq 0$.

(52) If $\operatorname{rng} q_1 \subseteq \operatorname{rng} p_1$ and $W$ is weight of $G$ and $i \in \operatorname{dom} q_1$, then there exists $j$ such that $j \in \operatorname{dom} p_1$ and $(\text{RealSequence}(p_1, W))(j) = (\text{RealSequence}(q_1, W))(i)$.

(53) If $\operatorname{len} q_1 = 1$ and $\operatorname{rng} q_1 \subseteq \operatorname{rng} p_1$ and $W$ is nonnegative weight of $G$, then $\text{cost}(q_1, W) \leq \text{cost}(p_1, W)$.

(54) If $W$ is nonnegative weight of $G$, then $\text{cost}(p_1, W) \geq 0$.

(55) If $p_1 = \emptyset$ and $W$ is weight of $G$, then $\text{cost}(p_1, W) = 0$.

(56) Let $D$ be a non empty finite subset of (the edges of $G$)$^*$. If $D = \text{AcyclicPaths}(v_1, v_2)$, then there exists $p_1$ such that $p_1 \in D$ and for every $q_1$ such that $q_1 \in D$ holds $\text{cost}(p_1, W) \leq \text{cost}(q_1, W)$.

(57) Let $D$ be a non empty finite subset of (the edges of $G$)$^*$. If $D = \text{AcyclicPaths}(v_1, v_2, V)$, then there exists $p_1$ such that $p_1 \in D$ and for every $q_1$ such that $q_1 \in D$ holds $\text{cost}(p_1, W) \leq \text{cost}(q_1, W)$.

(58) If $W$ is weight of $G$, then $\text{cost}(p_1 \frown q_1, W) = \text{cost}(p_1, W) + \text{cost}(q_1, W)$.

(59) If $q_1$ is one-to-one and $\operatorname{rng} q_1 \subseteq \operatorname{rng} p_1$ and $W$ is nonnegative weight of $G$, then $\text{cost}(q_1, W) \leq \text{cost}(p_1, W)$.

(60) If $p_1 \in \text{AcyclicPaths}(p)$ and $W$ is nonnegative weight of $G$, then $\text{cost}(p_1, W) \leq \text{cost}(p, W)$.

## 9. Shortest Paths and Their Basic Properties

Let $G$ be a graph, let $v_1$, $v_2$ be vertices of $G$, let $p$ be an oriented chain of $G$, and let $W$ be a function. We say that $p$ is shortest path from $v_1$ to $v_2$ in $W$ if and only if the conditions (Def. 17) are satisfied.

(Def. 17)(i) $p$ is oriented path from $v_1$ to $v_2$, and

(ii) for every oriented chain $q$ of $G$ such that $q$ is oriented path from $v_1$ to $v_2$ holds $\text{cost}(p, W) \leq \text{cost}(q, W)$.

Let $G$ be a graph, let $v_1$, $v_2$ be vertices of $G$, let $p$ be an oriented chain of $G$, let $V$ be a set, and let $W$ be a function. We say that $p$ is shortest path from $v_1$ to $v_2$ in $V$ w.r.t. $W$ if and only if the conditions (Def. 18) are satisfied.

(Def. 18)(i) $p$ is oriented path from $v_1$ to $v_2$ in $V$, and

(ii) for every oriented chain $q$ of $G$ such that $q$ is oriented path from $v_1$ to $v_2$ in $V$ holds $\text{cost}(p, W) \leq \text{cost}(q, W)$.

## 10. Basic Properties of a Graph with Finite Vertices

For simplicity, we use the following convention: $G$ is a finite graph, $p_4$ is a Simple oriented chain of $G$, $P$, $Q$ are oriented chains of $G$, $v_1$, $v_2$, $v_3$ are elements of the vertices of $G$, and $p_1$, $q_1$ are finite sequences of elements of the edges of $G$.

Next we state two propositions:

(61) $\mathrm{len}\, p_4 \leq$ the number of vertices of $G$.

(62) $\mathrm{len}\, p_4 \leq$ the number of edges of $G$.

Let us consider $G$. Observe that AcyclicPaths$(G)$ is finite.
Let us consider $G$, $P$. Note that AcyclicPaths$(P)$ is finite.
Let us consider $G$, $v_1$, $v_2$. One can check that AcyclicPaths$(v_1, v_2)$ is finite.
Let us consider $G$, $v_1$, $v_2$, $V$. Observe that AcyclicPaths$(v_1, v_2, V)$ is finite.
One can prove the following four propositions:

(63) If AcyclicPaths$(v_1, v_2) \neq \emptyset$, then there exists $p_1$ such that $p_1 \in$ AcyclicPaths$(v_1, v_2)$ and for every $q_1$ such that $q_1 \in$ AcyclicPaths$(v_1, v_2)$ holds cost$(p_1, W) \leq$ cost$(q_1, W)$.

(64) If AcyclicPaths$(v_1, v_2, V) \neq \emptyset$, then there exists $p_1$ such that $p_1 \in$ AcyclicPaths$(v_1, v_2, V)$ and for every $q_1$ such that $q_1 \in$ AcyclicPaths$(v_1, v_2, V)$ holds cost$(p_1, W) \leq$ cost$(q_1, W)$.

(65) If $P$ is oriented path from $v_1$ to $v_2$ and $W$ is nonnegative weight of $G$, then there exists a Simple oriented chain of $G$ which is shortest path from $v_1$ to $v_2$ in $W$.

(66) Suppose $P$ is oriented path from $v_1$ to $v_2$ in $V$ and $W$ is nonnegative weight of $G$. Then there exists a Simple oriented chain of $G$ which is shortest path from $v_1$ to $v_2$ in $V$ w.r.t. $W$.

## 11. Three Basic Theorems for Dijkstra's Shortest Path Algorithm

We now state two propositions:

(67) Suppose that

(i) $W$ is nonnegative weight of $G$,

(ii) $P$ is shortest path from $v_1$ to $v_2$ in $V$ w.r.t. $W$,

(iii) $v_1 \neq v_2$, and

(iv) for all $Q$, $v_3$ such that $v_3 \notin V$ and $Q$ is shortest path from $v_1$ to $v_3$ in $V$ w.r.t. $W$ holds cost$(P, W) \leq$ cost$(Q, W)$.

Then $P$ is shortest path from $v_1$ to $v_2$ in $W$.

(68) Suppose that

(i) $W$ is nonnegative weight of $G$,

(ii) $P$ is shortest path from $v_1$ to $v_2$ in $V$ w.r.t. $W$,

(iii) $v_1 \neq v_2$,

(iv) $V \subseteq U$, and

(v) for all $Q$, $v_3$ such that $v_3 \notin V$ and $Q$ is shortest path from $v_1$ to $v_3$ in $V$ w.r.t. $W$ holds cost$(P, W) \leq$ cost$(Q, W)$.

Then $P$ is shortest path from $v_1$ to $v_2$ in $U$ w.r.t. $W$.

Let $G$ be a graph, let $p_1$ be a finite sequence of elements of the edges of $G$, let $V$ be a set, let $v_1$ be a vertex of $G$, and let $W$ be a function. We say that $p_1$ is longest in shortest path from $v_1$ in $V$ w.r.t. $W$ if and only if the condition (Def. 19) is satisfied.

(Def. 19) Let $v$ be a vertex of $G$. Suppose $v \in V$ and $v \neq v_1$. Then there exists an oriented chain $q$ of $G$ such that $q$ is shortest path from $v_1$ to $v$ in $V$ w.r.t. $W$ and cost$(q, W) \leq$ cost$(p_1, W)$.

One can prove the following proposition

(69) Let $G$ be a finite oriented graph, $P$, $Q$, $R$ be oriented chains of $G$, and $v_1$, $v_2$, $v_3$ be elements of the vertices of $G$ such that $e \in$ the edges of $G$ and $W$ is nonnegative weight of $G$ and $\operatorname{len} P \geq 1$ and $P$ is shortest path from $v_1$ to $v_2$ in $V$ w.r.t. $W$ and $v_1 \neq v_2$ and $v_1 \neq v_3$ and $R = P \frown \langle e \rangle$ and $Q$ is shortest path from $v_1$ to $v_3$ in $V$ w.r.t. $W$ and $e$ orientedly joins $v_2$, $v_3$ and $P$ is longest in shortest path from $v_1$ in $V$ w.r.t. $W$. Then

(i) if $\mathrm{cost}(Q,W) \leq \mathrm{cost}(R,W)$, then $Q$ is shortest path from $v_1$ to $v_3$ in $V \cup \{v_2\}$ w.r.t. $W$, and

(ii) if $\mathrm{cost}(Q,W) \geq \mathrm{cost}(R,W)$, then $R$ is shortest path from $v_1$ to $v_3$ in $V \cup \{v_2\}$ w.r.t. $W$.

## REFERENCES


[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[6] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_2.html`.

[7] Czesław Byliński. The sum and product of finite sequences of real numbers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rvsum_1.html`.

[8] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[10] Krzysztof Hryniewiecki. Graphs. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/graph_1.html`.

[11] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-Board — part I. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/goboard1.html`.

[12] Yatsuka Nakamura and Piotr Rudnicki. Oriented chains. *Journal of Formalized Mathematics*, 10, 1998. `http://mizar.org/JFM/Vol10/graph_4.html`.

[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[14] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[15] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_4.html`.

[16] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[17] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.


*Received January 7, 2003*

*Published January 2, 2004*