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Euler Circuits and Paths[1]

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Summary. We prove the Euler theorem on existence of Euler circuits and paths in multigraphs.

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The articles [16], [7], [20], [17], [21], [4], [6], [5], [8], [10], [2], [18], [12], [3], [9], [19], [1], [13], [14], [11], and [15] provide the notation and terminology for this paper.

1. PRELIMINARIES

Let D be a set, let T be a non empty set of finite sequences of D, and let S be a non empty subset of T. We see that the element of S is a finite sequence of elements of D.

Let i, j be even integers. One can verify that $i-j$ is even.

One can prove the following propositions:

(1) For all integers i, j holds i is even iff j is even iff $i-j$ is even.

(2) Let p be a finite sequence and m, n, a be natural numbers. Suppose $a \in \mathrm{dom}\langle p(m),\ldots,p(n)\rangle$. Then there exists a natural number k such that $k \in \mathrm{dom}\, p$ and $p(k) = \langle p(m),\ldots,p(n)\rangle(a)$ and $k+1 = m+a$ and $m \leq k$ and $k \leq n$.

Let G be a graph. A vertex of G is an element of the vertices of G.

For simplicity, we adopt the following rules: G is a graph, v, v_1, v_2 are vertices of G, c is a chain of G, p, p_1, p_2 are paths of G, v_3, v_4, v_5 are finite sequences of elements of the vertices of G, e, X are sets, and n, m are natural numbers.

The following propositions are true:

(3) If v_3 is vertex sequence of c, then v_3 is non empty.

(7)[1] If $e \in$ the edges of G, then $\langle e\rangle$ is a path of G.

(8) $\langle p(m),\ldots,p(n)\rangle$ is a path of G.

(9) Suppose $\mathrm{rng}\, p_1$ misses $\mathrm{rng}\, p_2$ and v_4 is vertex sequence of p_1 and v_5 is vertex sequence of p_2 and $v_4(\mathrm{len}\, v_4) = v_5(1)$. Then $p_1 \frown p_2$ is a path of G.

(12)[2] If $c = \emptyset$, then c is cyclic.

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[1] The propositions (4)–(6) have been removed.

[2] The propositions (10) and (11) have been removed.

Let G be a graph. Observe that there exists a path of G which is cyclic.
Next we state several propositions:

(13) For every cyclic path p of G holds $\langle p(m+1),\ldots,p(\operatorname{len} p)\rangle \frown \langle p(1),\ldots,p(m)\rangle$ is a cyclic path of G.

(14) If $m+1 \in \operatorname{dom} p$, then $\operatorname{len}(\langle p(m+1),\ldots,p(\operatorname{len} p)\rangle \frown \langle p(1),\ldots,p(m)\rangle) = \operatorname{len} p$ and $\operatorname{rng}(\langle p(m+1),\ldots,p(\operatorname{len} p)\rangle \frown \langle p(1),\ldots,p(m)\rangle) = \operatorname{rng} p$ and $(\langle p(m+1),\ldots,p(\operatorname{len} p)\rangle \frown \langle p(1),\ldots,p(m)\rangle)(1) = p(m+1)$.

(15) For every cyclic path p of G such that $n \in \operatorname{dom} p$ there exists a cyclic path p' of G such that $p'(1) = p(n)$ and $\operatorname{len} p' = \operatorname{len} p$ and $\operatorname{rng} p' = \operatorname{rng} p$.

(16) Let s, t be vertices of G. Suppose $s =$ (the source of $G)(e)$ and $t =$ (the target of $G)(e)$. Then $\langle t,s\rangle$ is vertex sequence of $\langle e\rangle$.

(17) Suppose $e \in$ the edges of G and v_3 is vertex sequence of c and $v_3(\operatorname{len} v_3) =$ (the source of $G)(e)$. Then

(i) $c \frown \langle e\rangle$ is a chain of G, and

(ii) there exists a finite sequence v'_1 of elements of the vertices of G such that $v'_1 = v_3 \frown\frown \langle$(the source of $G)(e)$,(the target of $G)(e)\rangle$ and v'_1 is vertex sequence of $c \frown \langle e\rangle$ and $v'_1(1) = v_3(1)$ and $v'_1(\operatorname{len} v'_1) =$ (the target of $G)(e)$.

(18) Suppose $e \in$ the edges of G and v_3 is vertex sequence of c and $v_3(\operatorname{len} v_3) =$ (the target of $G)(e)$. Then

(i) $c \frown \langle e\rangle$ is a chain of G, and

(ii) there exists a finite sequence v'_1 of elements of the vertices of G such that $v'_1 = v_3 \frown\frown \langle$(the target of $G)(e)$,(the source of $G)(e)\rangle$ and v'_1 is vertex sequence of $c \frown \langle e\rangle$ and $v'_1(1) = v_3(1)$ and $v'_1(\operatorname{len} v'_1) =$ (the source of $G)(e)$.

(19) Suppose v_3 is vertex sequence of c. Let n be a natural number. Suppose $n \in \operatorname{dom} c$. Then

(i) $v_3(n) =$ (the target of $G)(c(n))$ and $v_3(n+1) =$ (the source of $G)(c(n))$, or

(ii) $v_3(n) =$ (the source of $G)(c(n))$ and $v_3(n+1) =$ (the target of $G)(c(n))$.

(20) If v_3 is vertex sequence of c and $e \in \operatorname{rng} c$, then (the target of $G)(e) \in \operatorname{rng} v_3$ and (the source of $G)(e) \in \operatorname{rng} v_3$.

Let G be a graph and let X be a set. Then G-VSet(X) is a subset of the vertices of G.
We now state four propositions:

(21) G-VSet$(\emptyset) = \emptyset$.

(22) If $e \in$ the edges of G and $e \in X$, then G-VSet(X) is non empty.

(23) G is connected if and only if for all v_1, v_2 such that $v_1 \neq v_2$ there exist c, v_3 such that c is non empty and v_3 is vertex sequence of c and $v_3(1) = v_1$ and $v_3(\operatorname{len} v_3) = v_2$.

(24) Let G be a connected graph, X be a set, and v be a vertex of G. Suppose X meets the edges of G and $v \notin G$-VSet(X). Then there exists a vertex v' of G and there exists an element e of the edges of G such that $v' \in G$-VSet(X) but $e \notin X$ but $v' =$ (the target of $G)(e)$ or $v' =$ (the source of $G)(e)$.

2. Degree of a vertex

Let G be a graph, let v be a vertex of G, and let X be a set. The functor $\mathrm{EdgesIn}(v,X)$ yields a subset of the edges of G and is defined as follows:

(Def. 1) For every set e holds $e \in \mathrm{EdgesIn}(v,X)$ iff $e \in$ the edges of G and $e \in X$ and (the target of $G)(e) = v$.

The functor $\mathrm{EdgesOut}(v,X)$ yields a subset of the edges of G and is defined as follows:

(Def. 2) For every set e holds $e \in \mathrm{EdgesOut}(v,X)$ iff $e \in$ the edges of G and $e \in X$ and (the source of $G)(e) = v$.

Let G be a graph, let v be a vertex of G, and let X be a set. The functor $\mathrm{EdgesAt}(v,X)$ yields a subset of the edges of G and is defined as follows:

(Def. 3) $\mathrm{EdgesAt}(v,X) = \mathrm{EdgesIn}(v,X) \cup \mathrm{EdgesOut}(v,X)$.

Let G be a finite graph, let v be a vertex of G, and let X be a set. One can check the following observations:

* $\mathrm{EdgesIn}(v,X)$ is finite,
* $\mathrm{EdgesOut}(v,X)$ is finite, and
* $\mathrm{EdgesAt}(v,X)$ is finite.

Let G be a graph, let v be a vertex of G, and let X be an empty set. One can check the following observations:

* $\mathrm{EdgesIn}(v,X)$ is empty,
* $\mathrm{EdgesOut}(v,X)$ is empty, and
* $\mathrm{EdgesAt}(v,X)$ is empty.

Let G be a graph and let v be a vertex of G. The functor $\mathrm{EdgesIn}\, v$ yields a subset of the edges of G and is defined as follows:

(Def. 4) $\mathrm{EdgesIn}\, v = \mathrm{EdgesIn}(v, \text{the edges of } G)$.

The functor $\mathrm{EdgesOut}\, v$ yields a subset of the edges of G and is defined by:

(Def. 5) $\mathrm{EdgesOut}\, v = \mathrm{EdgesOut}(v, \text{the edges of } G)$.

We now state two propositions:

(25) $\mathrm{EdgesIn}(v,X) \subseteq \mathrm{EdgesIn}\, v$.

(26) $\mathrm{EdgesOut}(v,X) \subseteq \mathrm{EdgesOut}\, v$.

Let G be a finite graph and let v be a vertex of G. Observe that $\mathrm{EdgesIn}\, v$ is finite and $\mathrm{EdgesOut}\, v$ is finite.

For simplicity, we adopt the following convention: G is a finite graph, v is a vertex of G, c is a chain of G, v_3 is a finite sequence of elements of the vertices of G, and X_1, X_2 are sets.

One can prove the following propositions:

(27) $\mathrm{card}\, \mathrm{EdgesIn}\, v = \mathrm{EdgIn}(v)$.

(28) $\mathrm{card}\, \mathrm{EdgesOut}\, v = \mathrm{EdgOut}(v)$.

Let G be a finite graph, let v be a vertex of G, and let X be a set. The functor $\mathrm{Degree}(v,X)$ yields a natural number and is defined as follows:

(Def. 6) $\mathrm{Degree}(v,X) = \mathrm{card}\, \mathrm{EdgesIn}(v,X) + \mathrm{card}\, \mathrm{EdgesOut}(v,X)$.

The following propositions are true:

(29) The degree of v = Degree(v, the edges of G).

(30) If Degree(v, X) $\neq 0$, then EdgesAt(v, X) is non empty.

(31) Suppose $e \in$ the edges of G but $e \notin X$ but v = (the target of G)(e) or v = (the source of G)(e). Then the degree of $v \neq$ Degree(v, X).

(32) If $X_2 \subseteq X_1$, then card EdgesIn($v, X_1 \setminus X_2$) = card EdgesIn(v, X_1) $-$ card EdgesIn(v, X_2).

(33) If $X_2 \subseteq X_1$, then card EdgesOut($v, X_1 \setminus X_2$) = card EdgesOut(v, X_1) $-$ card EdgesOut(v, X_2).

(34) If $X_2 \subseteq X_1$, then Degree($v, X_1 \setminus X_2$) = Degree(v, X_1) $-$ Degree(v, X_2).

(35) EdgesIn(v, X) = EdgesIn($v, X \cap$ the edges of G) and EdgesOut(v, X) = EdgesOut($v, X \cap$ the edges of G).

(36) Degree(v, X) = Degree($v, X \cap$ the edges of G).

(37) If c is non empty and v_3 is vertex sequence of c, then $v \in \operatorname{rng} v_3$ iff Degree(v, rng c) $\neq 0$.

(38) For every non empty finite connected graph G and for every vertex v of G holds the degree of $v \neq 0$.

3. Adding an Edge to a Graph

Let G be a graph and let v_1, v_2 be vertices of G. The functor AddNewEdge(v_1, v_2) yields a strict graph and is defined by the conditions (Def. 7).

(Def. 7)(i) The vertices of AddNewEdge(v_1, v_2) = the vertices of G,

(ii) the edges of AddNewEdge(v_1, v_2) = (the edges of G) $\cup$ {the edges of G},

(iii) the source of AddNewEdge(v_1, v_2) = (the source of G) $+\cdot$ ((the edges of G) $\dot{\longmapsto}$ (v_1)), and

(iv) the target of AddNewEdge(v_1, v_2) = (the target of G) $+\cdot$ ((the edges of G) $\dot{\longmapsto}$ (v_2)).

Let G be a finite graph and let v_1, v_2 be vertices of G. Observe that AddNewEdge(v_1, v_2) is finite.

For simplicity, we adopt the following convention: G denotes a graph, v, v_1, v_2 denote vertices of G, c denotes a chain of G, p denotes a path of G, v_3 denotes a finite sequence of elements of the vertices of G, v' denotes a vertex of AddNewEdge(v_1, v_2), p' denotes a path of AddNewEdge(v_1, v_2), and v'_1 denotes a finite sequence of elements of the vertices of AddNewEdge(v_1, v_2).

The following propositions are true:

(39)(i) The edges of $G \in$ the edges of AddNewEdge(v_1, v_2),

(ii) the edges of G = (the edges of AddNewEdge(v_1, v_2)) $\setminus$ {the edges of G},

(iii) (the source of AddNewEdge(v_1, v_2))(the edges of G) = v_1, and

(iv) (the target of AddNewEdge(v_1, v_2))(the edges of G) = v_2.

(40) Suppose $e \in$ the edges of G. Then (the source of AddNewEdge(v_1, v_2))(e) = (the source of G)(e) and (the target of AddNewEdge(v_1, v_2))(e) = (the target of G)(e).

(41) If $v'_1 = v_3$ and v_3 is vertex sequence of c, then v'_1 is vertex sequence of c.

(42) c is a chain of AddNewEdge(v_1, v_2).

(43) p is a path of AddNewEdge(v_1, v_2).

(44) If $v' = v_1$ and $v_1 \neq v_2$, then EdgesIn(v', X) = EdgesIn(v_1, X).

(45) If $v' = v_2$ and $v_1 \neq v_2$, then EdgesOut(v', X) = EdgesOut(v_2, X).

(46) If $v' = v_1$ and $v_1 \neq v_2$ and the edges of $G \in X$, then $\mathrm{EdgesOut}(v', X) = \mathrm{EdgesOut}(v_1, X) \cup$ {the edges of G} and $\mathrm{EdgesOut}(v_1, X)$ misses {the edges of G}.

(47) If $v' = v_2$ and $v_1 \neq v_2$ and the edges of $G \in X$, then $\mathrm{EdgesIn}(v', X) = \mathrm{EdgesIn}(v_2, X) \cup$ {the edges of G} and $\mathrm{EdgesIn}(v_2, X)$ misses {the edges of G}.

(48) If $v' = v$ and $v \neq v_1$ and $v \neq v_2$, then $\mathrm{EdgesIn}(v', X) = \mathrm{EdgesIn}(v, X)$.

(49) If $v' = v$ and $v \neq v_1$ and $v \neq v_2$, then $\mathrm{EdgesOut}(v', X) = \mathrm{EdgesOut}(v, X)$.

(50) If the edges of $G \notin \mathrm{rng}\, p'$, then p' is a path of G.

(51) If the edges of $G \notin \mathrm{rng}\, p'$ and $v_3 = v'_1$ and v'_1 is vertex sequence of p', then v_3 is vertex sequence of p'.

Let G be a connected graph and let v_1, v_2 be vertices of G. Observe that $\mathrm{AddNewEdge}(v_1, v_2)$ is connected.

For simplicity, we use the following convention: G is a finite graph, v, v_1, v_2 are vertices of G, v_3 is a finite sequence of elements of the vertices of G, and v' is a vertex of $\mathrm{AddNewEdge}(v_1, v_2)$.

The following two propositions are true:

(52) If $v' = v$ and $v_1 \neq v_2$ and $v = v_1$ or $v = v_2$ and the edges of $G \in X$, then $\mathrm{Degree}(v', X) = \mathrm{Degree}(v, X) + 1$.

(53) If $v' = v$ and $v \neq v_1$ and $v \neq v_2$, then $\mathrm{Degree}(v', X) = \mathrm{Degree}(v, X)$.

4. Some properties of and operations on cycles

We now state two propositions:

(54) For every cyclic path c of G holds $\mathrm{Degree}(v, \mathrm{rng}\, c)$ is even.

(55) Let c be a path of G. Suppose c is non cyclic and v_3 is vertex sequence of c. Then $\mathrm{Degree}(v, \mathrm{rng}\, c)$ is even if and only if $v \neq v_3(1)$ and $v \neq v_3(\mathrm{len}\, v_3)$.

In the sequel G is a graph, v is a vertex of G, and v_3 is a finite sequence of elements of the vertices of G.

Let G be a graph. The functor G-CycleSet yields a non empty set of finite sequences of the edges of G and is defined by:

(Def. 8) For every set x holds $x \in G$-CycleSet iff x is a cyclic path of G.

One can prove the following two propositions:

(56) $\emptyset$ is an element of G-CycleSet.

(57) Let c be an element of G-CycleSet. Suppose $v \in G\text{-VSet}(\mathrm{rng}\, c)$. Then $\{c'; c'$ ranges over elements of G-CycleSet: $\mathrm{rng}\, c' = \mathrm{rng}\, c \wedge \bigvee_{v_3} (v_3$ is vertex sequence of $c' \wedge v_3(1) = v)\}$ is a non empty subset of G-CycleSet.

Let us consider G, v and let c be an element of G-CycleSet. Let us assume that $v \in G\text{-VSet}(\mathrm{rng}\, c)$. The functor $c \circlearrowleft v$ yielding an element of G-CycleSet is defined as follows:

(Def. 9) $c \circlearrowleft v = \mathrm{choose}(\{c'; c'$ ranges over elements of G-CycleSet: $\mathrm{rng}\, c' = \mathrm{rng}\, c \wedge \bigvee_{v_3} (v_3$ is vertex sequence of $c' \wedge v_3(1) = v)\})$.

Let G be a graph and let c_1, c_2 be elements of G-CycleSet. Let us assume that $G\text{-VSet}(\mathrm{rng}\, c_1)$ meets $G\text{-VSet}(\mathrm{rng}\, c_2)$ and $\mathrm{rng}\, c_1$ misses $\mathrm{rng}\, c_2$. The functor $\mathrm{CatCycles}(c_1, c_2)$ yields an element of G-CycleSet and is defined as follows:

(Def. 10) There exists a vertex v of G such that $v = \mathrm{choose}((G\text{-VSet}(\mathrm{rng}\, c_1)) \cap (G\text{-VSet}(\mathrm{rng}\, c_2)))$ and $\mathrm{CatCycles}(c_1, c_2) = (c_1 \circlearrowleft v) ^\frown (c_2 \circlearrowleft v)$.

The following proposition is true

(58) Let G be a graph and c_1, c_2 be elements of G-CycleSet. Suppose G-VSet(rng c_1) meets G-VSet(rng c_2) but rng c_1 misses rng c_2 but $c_1 \neq \emptyset$ or $c_2 \neq \emptyset$. Then CatCycles(c_1, c_2) is non empty.

In the sequel G is a finite graph, v is a vertex of G, and v_3 is a finite sequence of elements of the vertices of G.

Let us consider G, v and let X be a set. Let us assume that Degree$(v,X) \neq 0$. The functor X-PathSet(v) yields a non empty set of finite sequences of the edges of G and is defined as follows:

(Def. 11) X-PathSet$(v) = \{c; c$ ranges over elements of X^*: c is a path of G $\wedge$ $c$is non empty $\wedge$ $\bigvee_{v_3}$ (v_3 is vertex sequence of c $\wedge$ $v_3(1) = v)\}$.

We now state the proposition

(59) For every element p of X-PathSet(v) and for every finite set Y such that $Y =$ the edges of G and Degree$(v,X) \neq 0$ holds len $p \leq$ card Y.

Let us consider G, v and let X be a set. Let us assume that for every vertex v of G holds Degree(v,X) is even and Degree$(v,X) \neq 0$. The functor X-CycleSet$v$ yields a non empty subset of G-CycleSet and is defined as follows:

(Def. 12) X-CycleSet$v = \{c; c$ ranges over elements of G-CycleSet: rng $c \subseteq X$ $\wedge$ $c$is non empty $\wedge$ $\bigvee_{v_3}$ (v_3 is vertex sequence of c $\wedge$ $v_3(1) = v)\}$.

The following two propositions are true:

(60) If Degree$(v,X) \neq 0$ and for every v holds Degree(v,X) is even, then for every element c of X-CycleSet$v$ holds c is non empty and rng $c \subseteq X$ and $v \in G$-VSet(rng c).

(61) Let G be a finite connected graph and c be an element of G-CycleSet. Suppose rng $c \neq$ the edges of G and c is non empty. Then $\{v'; v'$ ranges over vertices of G: $v' \in G$-VSet(rng c) $\wedge$ the degree of $v' \neq$ Degree$(v', \text{rng}\, c)\}$ is a non empty subset of the vertices of G.

Let G be a finite connected graph and let c be an element of G-CycleSet. Let us assume that rng $c \neq$ the edges of G and c is non empty. The functor ExtendCycle c yields an element of G-CycleSet and is defined by the condition (Def. 13).

(Def. 13) There exists an element c' of G-CycleSet and there exists a vertex v of G such that $v =$ choose$(\{v'; v'$ ranges over vertices of G: $v' \in G$-VSet(rng c) $\wedge$ the degree of $v' \neq$ Degree$(v', \text{rng}\, c)\})$ and $c' =$ choose(((the edges of G) $\setminus$ rng c)-CycleSet$v$) and ExtendCycle $c =$ CatCycles(c, c').

Next we state the proposition

(62) Let G be a finite connected graph and c be an element of G-CycleSet. Suppose rng $c \neq$ the edges of G and c is non empty and for every vertex v of G holds the degree of v is even. Then ExtendCycle c is non empty and card rng $c <$ card rng ExtendCycle c.

5. Euler circuits and paths

Let G be a graph and let p be a path of G. We say that p is Eulerian if and only if:

(Def. 14) rng $p =$ the edges of G.

One can prove the following three propositions:

(63) Let G be a connected graph, p be a path of G, and v_3 be a finite sequence of elements of the vertices of G. Suppose p is Eulerian and v_3 is vertex sequence of p. Then rng $v_3 =$ the vertices of G.

(64) Let G be a finite connected graph. Then there exists a cyclic path of G which is Eulerian if and only if for every vertex v of G holds the degree of v is even.

(65) Let G be a finite connected graph. Then there exists a path of G which is non cyclic and Eulerian if and only if there exist vertices v_1, v_2 of G such that $v_1 \neq v_2$ and for every vertex v of G holds the degree of v is even iff $v \neq v_1$ and $v \neq v_2$.

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