JOURNAL OF FORMALIZED MATHEMATICS
Volume 7, Released 1995, Published 2003
Inst. of Computer Science, Univ. of Białystok

# Vertex Sequences Induced by Chains[1]

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**Summary.** In the three preliminary sections to the article we define two operations on finite sequences which seem to be of general interest. The first is the *cut* operation that extracts a contiguous chunk of a finite sequence from a position to a position. The second operation is a glueing catenation that given two finite sequences catenates them with removal of the first element of the second sequence. The main topic of the article is to define an operation which for a given chain in a graph returns the sequence of vertices through which the chain passes. We define the exact conditions when such an operation is uniquely definable. This is done with the help of the so called two-valued alternating finite sequences. We also prove theorems about the existence of simple chains which are subchains of a given chain. In order to do this we define the notion of a finite subsequence of a typed finite sequence.

MML Identifier: `GRAPH_2`.

WWW: `http://mizar.org/JFM/Vol7/graph_2.html`

The articles [10], [13], [11], [14], [4], [5], [3], [6], [12], [1], [7], [2], [8], and [9] provide the notation and terminology for this paper.

## 1. Preliminaries

We use the following convention: $p$, $q$ denote finite sequences, $X$, $Y$ denote sets, and $i$, $k$, $l$, $m$, $n$, $r$ denote natural numbers.

The scheme *FinSegRng* deals with natural numbers $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(i) : \mathcal{A} \leq i \wedge i \leq \mathcal{B} \wedge \mathcal{P}[i]\}$ is finite

for all values of the parameters.

One can prove the following propositions:

(1) $m+1 \leq k$ and $k \leq n$ iff there exists a natural number $i$ such that $m \leq i$ and $i < n$ and $k = i+1$.

(2) If $q = p \restriction \mathrm{Seg}\, n$, then $\mathrm{len}\, q \leq \mathrm{len}\, p$ and for every $i$ such that $1 \leq i$ and $i \leq \mathrm{len}\, q$ holds $p(i) = q(i)$.

(3) If $X \subseteq \mathrm{Seg}\, k$ and $Y \subseteq \mathrm{dom}\, \mathrm{Sgm}\, X$, then $\mathrm{Sgm}\, X \cdot \mathrm{Sgm}\, Y = \mathrm{Sgm}\, \mathrm{rng}(\mathrm{Sgm}\, X \restriction Y)$.

(4) For all natural numbers $m$, $n$ holds $\overline{\overline{\{k : m \leq k \wedge k \leq m+n\}}} = n+1$.

(5) For every $l$ such that $1 \leq l$ and $l \leq n$ holds $(\mathrm{Sgm}\{k_1; k_1$ ranges over natural numbers: $m+1 \leq k_1 \wedge k_1 \leq m+n\})(l) = m+l$.

[1]This work was partially supported by Shinshu Endowment for Information Science, NSERC Grant OGP9207 and JSTF award 651-93-S009.


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## 2. THE CUT OPERATION FOR FINITE SEQUENCES

Let $p$ be a finite sequence and let $m$, $n$ be natural numbers. The functor $\langle p(m),\ldots,p(n)\rangle$ yields a finite sequence and is defined by:

(Def. 1)(i) $\mathrm{len}\langle p(m),\ldots,p(n)\rangle + m = n+1$ and for every natural number $i$ such that $i < \mathrm{len}\langle p(m),\ldots,p(n)\rangle$ holds $\langle p(m),\ldots,p(n)\rangle(i+1) = p(m+i)$ if $1 \le m$ and $m \le n+1$ and $n \le \mathrm{len}\,p$,

(ii) $\langle p(m),\ldots,p(n)\rangle = \emptyset$, otherwise.

We now state several propositions:

(6) If $1 \le m$ and $m \le \mathrm{len}\,p$, then $\langle p(m),\ldots,p(m)\rangle = \langle p(m)\rangle$.

(7) $\langle p(1),\ldots,p(\mathrm{len}\,p)\rangle = p$.

(8) If $m \le n$ and $n \le r$ and $r \le \mathrm{len}\,p$, then $\langle p(m+1),\ldots,p(n)\rangle \frown \langle p(n+1),\ldots,p(r)\rangle = \langle p(m+1),\ldots,p(r)\rangle$.

(9) If $m \le \mathrm{len}\,p$, then $\langle p(1),\ldots,p(m)\rangle \frown \langle p(m+1),\ldots,p(\mathrm{len}\,p)\rangle = p$.

(10) If $m \le n$ and $n \le \mathrm{len}\,p$, then $\langle p(1),\ldots,p(m)\rangle \frown \langle p(m+1),\ldots,p(n)\rangle \frown \langle p(n+1),\ldots,p(\mathrm{len}\,p)\rangle = p$.

(11) $\mathrm{rng}\langle p(m),\ldots,p(n)\rangle \subseteq \mathrm{rng}\,p$.

Let $D$ be a set, let $p$ be a finite sequence of elements of $D$, and let $m$, $n$ be natural numbers. Then $\langle p(m),\ldots,p(n)\rangle$ is a finite sequence of elements of $D$.

We now state the proposition

(12) If $1 \le m$ and $m \le n$ and $n \le \mathrm{len}\,p$, then $\langle p(m),\ldots,p(n)\rangle(1) = p(m)$ and $\langle p(m),\ldots,p(n)\rangle(\mathrm{len}\langle p(m),\ldots,p(n)\rangle) = p(n)$.

## 3. THE GLUEING CATENATION OF FINITE SEQUENCES

Let $p$, $q$ be finite sequences. The functor $p \frown\!\!\frown q$ yields a finite sequence and is defined as follows:

(Def. 2) $p \frown\!\!\frown q = p \frown \langle q(2),\ldots,q(\mathrm{len}\,q)\rangle$.

Next we state several propositions:

(13) If $q \neq \emptyset$, then $\mathrm{len}(p \frown\!\!\frown q) + 1 = \mathrm{len}\,p + \mathrm{len}\,q$.

(14) If $1 \le k$ and $k \le \mathrm{len}\,p$, then $(p \frown\!\!\frown q)(k) = p(k)$.

(15) If $1 \le k$ and $k < \mathrm{len}\,q$, then $(p \frown\!\!\frown q)(\mathrm{len}\,p + k) = q(k+1)$.

(16) If $1 < \mathrm{len}\,q$, then $(p \frown\!\!\frown q)(\mathrm{len}(p \frown\!\!\frown q)) = q(\mathrm{len}\,q)$.

(17) $\mathrm{rng}(p \frown\!\!\frown q) \subseteq \mathrm{rng}\,p \cup \mathrm{rng}\,q$.

Let $D$ be a set and let $p$, $q$ be finite sequences of elements of $D$. Then $p \frown\!\!\frown q$ is a finite sequence of elements of $D$.

We now state the proposition

(18) If $p \neq \emptyset$ and $q \neq \emptyset$ and $p(\mathrm{len}\,p) = q(1)$, then $\mathrm{rng}(p \frown\!\!\frown q) = \mathrm{rng}\,p \cup \mathrm{rng}\,q$.

## 4. TWO VALUED ALTERNATING FINITE SEQUENCES

Let $f$ be a finite sequence. We say that $f$ is two-valued if and only if:

(Def. 3) $\operatorname{card}\operatorname{rng} f = 2$.

Next we state the proposition

(19) $p$ is two-valued iff $\operatorname{len} p > 1$ and there exist sets $x$, $y$ such that $x \neq y$ and $\operatorname{rng} p = \{x, y\}$.

Let $f$ be a finite sequence. We say that $f$ is alternating if and only if:

(Def. 4) For every natural number $i$ such that $1 \leq i$ and $i+1 \leq \operatorname{len} f$ holds $f(i) \neq f(i+1)$.

Let us note that there exists a finite sequence which is two-valued and alternating.
In the sequel $a$, $a_1$, $a_2$ denote two-valued alternating finite sequences.
We now state four propositions:

(20) If $\operatorname{len} a_1 = \operatorname{len} a_2$ and $\operatorname{rng} a_1 = \operatorname{rng} a_2$ and $a_1(1) = a_2(1)$, then $a_1 = a_2$.

(21) If $a_1 \neq a_2$ and $\operatorname{len} a_1 = \operatorname{len} a_2$ and $\operatorname{rng} a_1 = \operatorname{rng} a_2$, then for every $i$ such that $1 \leq i$ and $i \leq \operatorname{len} a_1$ holds $a_1(i) \neq a_2(i)$.

(22) If $a_1 \neq a_2$ and $\operatorname{len} a_1 = \operatorname{len} a_2$ and $\operatorname{rng} a_1 = \operatorname{rng} a_2$, then for every $a$ such that $\operatorname{len} a = \operatorname{len} a_1$ and $\operatorname{rng} a = \operatorname{rng} a_1$ holds $a = a_1$ or $a = a_2$.

(23) If $X \neq Y$ and $n > 1$, then there exists $a_1$ such that $\operatorname{rng} a_1 = \{X, Y\}$ and $\operatorname{len} a_1 = n$ and $a_1(1) = X$.

## 5. FINITE SUBSEQUENCE OF FINITE SEQUENCES

Let us consider $X$ and let $f_1$ be a finite sequence of elements of $X$. A finite subsequence is said to be a FinSubsequence of $f_1$ if:

(Def. 5) It $\subseteq f_1$.

In the sequel $s_1$ denotes a finite subsequence.
The following two propositions are true:

(24) If $s_1$ is a finite sequence, then $\operatorname{Seq} s_1 = s_1$.

(26)[1] Let $f$ be a finite subsequence and $g$, $h$, $f_2$, $f_3$, $f_4$ be finite sequences. If $\operatorname{rng} g \subseteq \operatorname{dom} f$ and $\operatorname{rng} h \subseteq \operatorname{dom} f$ and $f_2 = f \cdot g$ and $f_3 = f \cdot h$ and $f_4 = f \cdot (g ^\frown h)$, then $f_4 = f_2 ^\frown f_3$.

We adopt the following rules: $f_1$, $f_5$, $f_6$ denote finite sequences of elements of $X$ and $f_7$, $f_8$ denote FinSubsequences of $f_1$.
One can prove the following four propositions:

(27) $\operatorname{dom} f_7 \subseteq \operatorname{dom} f_1$ and $\operatorname{rng} f_7 \subseteq \operatorname{rng} f_1$.

(28) $f_1$ is a FinSubsequence of $f_1$.

(29) $f_7 \restriction Y$ is a FinSubsequence of $f_1$.

(30) For every FinSubsequence $f_9$ of $f_5$ such that $\operatorname{Seq} f_7 = f_5$ and $\operatorname{Seq} f_9 = f_6$ and $f_8 = f_7 \restriction \operatorname{rng}(\operatorname{Sgm} \operatorname{dom} f_7 \restriction \operatorname{dom} f_9)$ holds $\operatorname{Seq} f_8 = f_6$.

[1] The proposition (25) has been removed.

## 6. VERTEX SEQUENCES INDUCED BY CHAINS

In the sequel $G$ is a graph.

Let us consider $G$. One can check that the vertices of $G$ is non empty.

We adopt the following rules: $v$, $v_1$, $v_2$, $v_3$, $v_4$ are elements of the vertices of $G$ and $e$ is a set.

One can prove the following two propositions:

(31) If $e$ joins $v_1$ with $v_2$, then $e$ joins $v_2$ with $v_1$.

(32) If $e$ joins $v_1$ with $v_2$ and $e$ joins $v_3$ with $v_4$, then $v_1 = v_3$ and $v_2 = v_4$ or $v_1 = v_4$ and $v_2 = v_3$.

We follow the rules: $v_5$, $v_6$, $v_7$ denote finite sequences of elements of the vertices of $G$ and $c$, $c_1$, $c_2$ denote chains of $G$.

Let us consider $G$. Note that there exists a chain of $G$ which is empty.

Let us consider $G$, $X$. The functor $G$-VSet$(X)$ yields a set and is defined by:

(Def. 6) $G\text{-VSet}(X) = \{v : \bigvee_{e:\text{element of the edges of }G} (e \in X \ \wedge \ (v = (\text{the source of } G)(e) \ \vee \ v = (\text{the target of } G)(e)))\}$.

Let us consider $G$, $v_5$ and let $c$ be a finite sequence. We say that $v_5$ is vertex sequence of $c$ if and only if:

(Def. 7) $\operatorname{len} v_5 = \operatorname{len} c + 1$ and for every $n$ such that $1 \leq n$ and $n \leq \operatorname{len} c$ holds $c(n)$ joins $(v_5)_n$ with $(v_5)_{n+1}$.

We now state four propositions:

(34)[2] If $c \neq \emptyset$ and $v_5$ is vertex sequence of $c$, then $G\text{-VSet}(\operatorname{rng} c) = \operatorname{rng} v_5$.

(35) $\langle v \rangle$ is vertex sequence of $\emptyset$.

(36) There exists $v_5$ which is vertex sequence of $c$.

(37) Suppose $c \neq \emptyset$ and $v_6$ is vertex sequence of $c$ and $v_7$ is vertex sequence of $c$ and $v_6 \neq v_7$. Then $v_6(1) \neq v_7(1)$ and for every $v_5$ such that $v_5$ is vertex sequence of $c$ holds $v_5 = v_6$ or $v_5 = v_7$.

Let us consider $G$ and let $c$ be a finite sequence. We say that $c$ alternates vertices in $G$ if and only if:

(Def. 8) $\operatorname{len} c \geq 1$ and $\overline{\overline{G\text{-VSet}(\operatorname{rng} c)}} = 2$ and for every $n$ such that $n \in \operatorname{dom} c$ holds (the source of $G)(c(n)) \neq$ (the target of $G)(c(n))$.

Next we state several propositions:

(38) If $c$ alternates vertices in $G$ and $v_5$ is vertex sequence of $c$, then for every $k$ such that $k \in \operatorname{dom} c$ holds $v_5(k) \neq v_5(k+1)$.

(39) Suppose $c$ alternates vertices in $G$ and $v_5$ is vertex sequence of $c$. Then $\operatorname{rng} v_5 = \{(\text{the source of } G)(c(1)), (\text{the target of } G)(c(1))\}$.

(40) Suppose $c$ alternates vertices in $G$ and $v_5$ is vertex sequence of $c$. Then $v_5$ is a two-valued alternating finite sequence.

(41) Suppose $c$ alternates vertices in $G$. Then there exist $v_6$, $v_7$ such that

(i) $v_6 \neq v_7$,

(ii) $v_6$ is vertex sequence of $c$,

(iii) $v_7$ is vertex sequence of $c$, and

(iv) for every $v_5$ such that $v_5$ is vertex sequence of $c$ holds $v_5 = v_6$ or $v_5 = v_7$.

[2] The proposition (33) has been removed.

(42) Suppose $v_5$ is vertex sequence of $c$. Then $\overline{\overline{\text{the vertices of } G}} = 1$ or $c \neq \emptyset$ and $c$ does not alternate vertices in $G$ if and only if for every $v_6$ such that $v_6$ is vertex sequence of $c$ holds $v_6 = v_5$.

Let us consider $G$, $c$. Let us assume that $\overline{\overline{\text{the vertices of } G}} = 1$ or $c \neq \emptyset$ and $c$ does not alternate vertices in $G$. The functor vertex-seq$(c)$ yielding a finite sequence of elements of the vertices of $G$ is defined as follows:

(Def. 9) vertex-seq$(c)$ is vertex sequence of $c$.

Next we state several propositions:

(43) If $v_5$ is vertex sequence of $c$ and $c_1 = c \restriction \mathrm{Seg}\, n$ and $v_6 = v_5 \restriction \mathrm{Seg}(n+1)$, then $v_6$ is vertex sequence of $c_1$.

(44) If $1 \leq m$ and $m \leq n$ and $n \leq \mathrm{len}\, c$ and $q = \langle c(m), \ldots, c(n) \rangle$, then $q$ is a chain of $G$.

(45) If $1 \leq m$ and $m \leq n$ and $n \leq \mathrm{len}\, c$ and $c_1 = \langle c(m), \ldots, c(n) \rangle$ and $v_5$ is vertex sequence of $c$ and $v_6 = \langle v_5(m), \ldots, v_5(n+1) \rangle$, then $v_6$ is vertex sequence of $c_1$.

(46) If $v_6$ is vertex sequence of $c_1$ and $v_7$ is vertex sequence of $c_2$ and $v_6(\mathrm{len}\, v_6) = v_7(1)$, then $c_1 \frown c_2$ is a chain of $G$.

(47) Suppose $v_6$ is vertex sequence of $c_1$ and $v_7$ is vertex sequence of $c_2$ and $v_6(\mathrm{len}\, v_6) = v_7(1)$ and $c = c_1 \frown c_2$ and $v_5 = v_6 \frown\!\frown v_7$. Then $v_5$ is vertex sequence of $c$.

## 7. Vertex sequences induced by simple chains, paths and ordered chains

Let us consider $G$ and let $I_1$ be a chain of $G$. We say that $I_1$ is simple if and only if:

(Def. 10) There exists $v_5$ such that $v_5$ is vertex sequence of $I_1$ and for all $n$, $m$ such that $1 \leq n$ and $n < m$ and $m \leq \mathrm{len}\, v_5$ and $v_5(n) = v_5(m)$ holds $n = 1$ and $m = \mathrm{len}\, v_5$.

Let us consider $G$. One can verify that there exists a chain of $G$ which is simple.
In the sequel $s_2$ denotes a simple chain of $G$.
One can prove the following propositions:

(49)[3] $s_2 \restriction \mathrm{Seg}\, n$ is a simple chain of $G$.

(50) If $2 < \mathrm{len}\, s_2$ and $v_6$ is vertex sequence of $s_2$ and $v_7$ is vertex sequence of $s_2$, then $v_6 = v_7$.

(51) If $v_5$ is vertex sequence of $s_2$, then for all $n$, $m$ such that $1 \leq n$ and $n < m$ and $m \leq \mathrm{len}\, v_5$ and $v_5(n) = v_5(m)$ holds $n = 1$ and $m = \mathrm{len}\, v_5$.

(52) Suppose $c$ is not a simple chain of $G$ and $v_5$ is vertex sequence of $c$. Then there exists a FinSubsequence $f_{10}$ of $c$ and there exists a FinSubsequence $f_{11}$ of $v_5$ and there exist $c_1$, $v_6$ such that $\mathrm{len}\, c_1 < \mathrm{len}\, c$ and $v_6$ is vertex sequence of $c_1$ and $\mathrm{len}\, v_6 < \mathrm{len}\, v_5$ and $v_5(1) = v_6(1)$ and $v_5(\mathrm{len}\, v_5) = v_6(\mathrm{len}\, v_6)$ and $\mathrm{Seq}\, f_{10} = c_1$ and $\mathrm{Seq}\, f_{11} = v_6$.

(53) Suppose $v_5$ is vertex sequence of $c$. Then there exists a FinSubsequence $f_{10}$ of $c$ and there exists a FinSubsequence $f_{11}$ of $v_5$ and there exist $s_2$, $v_6$ such that $\mathrm{Seq}\, f_{10} = s_2$ and $\mathrm{Seq}\, f_{11} = v_6$ and $v_6$ is vertex sequence of $s_2$ and $v_5(1) = v_6(1)$ and $v_5(\mathrm{len}\, v_5) = v_6(\mathrm{len}\, v_6)$.

Let us consider $G$. One can verify that every chain of $G$ which is empty is also one-to-one.
Next we state the proposition

(54) If $p$ is a path of $G$, then $p \restriction \mathrm{Seg}\, n$ is a path of $G$.

Let us consider $G$. Observe that there exists a path of $G$ which is simple.
One can prove the following two propositions:

[3] The proposition (48) has been removed.

(55) If $2 < \operatorname{len} s_2$, then $s_2$ is a path of $G$.

(56) $s_2$ is a path of $G$ iff $\operatorname{len} s_2 = 0$ or $\operatorname{len} s_2 = 1$ or $s_2(1) \neq s_2(2)$.

Let us consider $G$. Observe that every chain of $G$ which is empty is also oriented.

Let us consider $G$ and let $o_1$ be an oriented chain of $G$. Let us assume that $o_1 \neq \emptyset$. The functor vertex-seq$(o_1)$ yielding a finite sequence of elements of the vertices of $G$ is defined as follows:

(Def. 11) vertex-seq$(o_1)$ is vertex sequence of $o_1$ and $(\text{vertex-seq}(o_1))(1) = (\text{the source of } G)(o_1(1))$.

## REFERENCES


[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[6] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_2.html`.

[7] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[8] Krzysztof Hryniewiecki. Graphs. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/graph_1.html`.

[9] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/binarith.html`.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[11] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[12] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_4.html`.

[13] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[14] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.


*Received May 13, 1995*

*Published January 2, 2004*