JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

A Construction of Analytical Ordered Trapezium Spaces[1]

Henryk Oryszczyszyn
Warsaw University
Białystok

Krzysztof Prażmowski
Warsaw University
Białystok

Summary. We define, in a given real linear space, the midpoint operation on vectors and, with the help of the notions of directed parallelism of vectors and orthogonality of vectors, we define the relation of directed trapezium. We consider structures being enrichments of affine structures by one binary operation, together with a function which assigns to every such a structure its "affine" reduct. Theorems concerning midpoint operation and trapezium relation are proved which enables us to introduce an abstract notion of (regular in fact) ordered trapezium space with midpoint, ordered trapezium space, and (unordered) trapezium space.

MML Identifier: GEOMTRAP.

WWW: http://mizar.org/JFM/Vol2/geomtrap.html

The articles [9], [2], [11], [8], [3], [7], [12], [10], [4], [6], [1], and [5] provide the notation and terminology for this paper.

For simplicity, we use the following convention: V denotes a real linear space, u, u_1, u_2, v, v_1, v_2, w, y denote vectors of V, a, b denote real numbers, and x, z denote sets.

Let us consider V and let us consider u, v, u_1, v_1. The predicate $u,v \parallel u_1,v_1$ is defined as follows:

(Def. 1) $u,v \upuparrows u_1,v_1$ or $u,v \upuparrows v_1,u_1$.

The following propositions are true:

(1) If w, y span the space, then OASpace V is an ordered affine space.

(2) For all elements p, q, p_1, q_1 of OASpace V such that $p = u$ and $q = v$ and $p_1 = u_1$ and $q_1 = v_1$ holds $p,q \upuparrows p_1,q_1$ iff $u,v \upuparrows u_1,v_1$.

(3) Suppose w, y span the space. Let p, q, p_1, q_1 be elements of the carrier of Λ(OASpace V). If $p = u$ and $q = v$ and $p_1 = u_1$ and $q_1 = v_1$, then $p,q \upuparrows p_1,q_1$ iff $u,v \parallel u_1,v_1$.

(4) Let p, q, p_1, q_1 be elements of the carrier of $\mathbf{AMSp}(V,w,y)$. If $p = u$ and $q = v$ and $p_1 = u_1$ and $q_1 = v_1$, then $p,q \upuparrows p_1,q_1$ iff $u,v \parallel u_1,v_1$.

Let us consider V and let us consider u, v. The functor $u\#v$ yields a vector of V and is defined as follows:

(Def. 2) $u\#v + u\#v = u + v$.

Let us notice that the functor $u\#v$ is commutative and idempotent.

We now state a number of propositions:

[1]Supported by RPBP.III-24.C2.

(7)[1] There exists y such that $u\#y = w$.

(8) $(u\#u_1)\#(v\#v_1) = u\#v\#(u_1\#v_1)$.

(9) If $u\#y = u\#w$, then $y = w$.

(10) $u, v \upuparrows y\#u, y\#v$.

(11) $u, v \parallel w\#u, w\#v$.

(12) $2 \cdot (u\#v - u) = v - u$ and $2 \cdot (v - u\#v) = v - u$.

(13) $u, u\#v \upuparrows u\#v, v$.

(14) $u, v \upuparrows u, u\#v$ and $u, v \upuparrows u\#v, v$.

(15) If $u, y \upuparrows y, v$, then $u\#y, y \upuparrows y, y\#v$.

(16) If $u, v \upuparrows u_1, v_1$, then $u, v \upuparrows u\#u_1, v\#v_1$.

Let us consider V and let us consider w, y, u, u_1, v, v_1. We say that u, u_1 and v, v_1 form a directed trapezium w.r.t. w, y if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $u, u_1 \upuparrows v, v_1$,

(ii) u, u_1, $u\#u_1$ and $v\#v_1$ are orthogonal w.r.t. w, y, and

(iii) v, v_1, $u\#u_1$ and $v\#v_1$ are orthogonal w.r.t. w, y.

One can prove the following propositions:

(17) If w, y span the space, then u, u and v, v form a directed trapezium w.r.t. w, y.

(18) If w, y span the space, then u, v and u, v form a directed trapezium w.r.t. w, y.

(19) If u, v and v, u form a directed trapezium w.r.t. w, y, then $u = v$.

(20) If w, y span the space and v_1, u and u, v_2 form a directed trapezium w.r.t. w, y, then $v_1 = u$ and $u = v_2$.

(21) Suppose that

(i) w, y span the space,

(ii) u, v and u_1, v_1 form a directed trapezium w.r.t. w, y,

(iii) u, v and u_2, v_2 form a directed trapezium w.r.t. w, y, and

(iv) $u \neq v$.

Then u_1, v_1 and u_2, v_2 form a directed trapezium w.r.t. w, y.

(22) Suppose w, y span the space. Then there exists a vector t of V such that

(i) u, v and u_1, t form a directed trapezium w.r.t. w, y, or

(ii) u, v and t, u_1 form a directed trapezium w.r.t. w, y.

(23) Suppose w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y. Then u_1, v_1 and u, v form a directed trapezium w.r.t. w, y.

(24) Suppose w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y. Then v, u and v_1, u_1 form a directed trapezium w.r.t. w, y.

(25) If w, y span the space and v, u_1 and v, u_2 form a directed trapezium w.r.t. w, y, then $u_1 = u_2$.

[1] The propositions (5) and (6) have been removed.

(26) Suppose that

(i) w, y span the space,

(ii) u, v and u_1, v_1 form a directed trapezium w.r.t. w, y, and

(iii) u, v and u_1, v_2 form a directed trapezium w.r.t. w, y.

Then $u = v$ or $v_1 = v_2$.

(27) Suppose that

(i) w, y span the space,

(ii) $u \neq u_1$,

(iii) u, u_1 and v, v_1 form a directed trapezium w.r.t. w, y, and

(iv) u, u_1 and v, v_2 form a directed trapezium w.r.t. w, y or u, u_1 and v_2, v form a directed trapezium w.r.t. w, y.

Then $v_1 = v_2$.

(28) Suppose w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y. Then u, v and $u\#u_1$, $v\#v_1$ form a directed trapezium w.r.t. w, y.

(29) Suppose w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y. Then

(i) u, v and $u\#v_1$, $v\#u_1$ form a directed trapezium w.r.t. w, y, or

(ii) u, v and $v\#u_1$, $u\#v_1$ form a directed trapezium w.r.t. w, y.

(30) Let u, u_1, u_2, v_1, v_2, t_1, t_2, w_1, w_2 be vectors of V. Suppose that

(i) w, y span the space,

(ii) $u = u_1\#t_1$,

(iii) $u = u_2\#t_2$,

(iv) $u = v_1\#w_1$,

(v) $u = v_2\#w_2$, and

(vi) u_1, u_2 and v_1, v_2 form a directed trapezium w.r.t. w, y.

Then t_1, t_2 and w_1, w_2 form a directed trapezium w.r.t. w, y.

Let us consider V, w, y, u. Let us assume that w, y span the space. The functor $\pi^1_{w,y}(u)$ yielding a real number is defined as follows:

(Def. 4) There exists b such that $u = \pi^1_{w,y}(u) \cdot w + b \cdot y$.

Let us consider V, w, y, u. Let us assume that w, y span the space. The functor $\pi^2_{w,y}(u)$ yielding a real number is defined as follows:

(Def. 5) There exists a such that $u = a \cdot w + \pi^2_{w,y}(u) \cdot y$.

Let us consider V, w, y, u, v. The functor $u \cdot_{w,y} v$ yields a real number and is defined as follows:

(Def. 6) $u \cdot_{w,y} v = \pi^1_{w,y}(u) \cdot \pi^1_{w,y}(v) + \pi^2_{w,y}(u) \cdot \pi^2_{w,y}(v)$.

We now state a number of propositions:

(31) For all u, v holds $u \cdot_{w,y} v = v \cdot_{w,y} u$.

(32) Suppose w, y span the space. Let given u, v, v_1. Then $u \cdot_{w,y} (v + v_1) = u \cdot_{w,y} v + u \cdot_{w,y} v_1$ and $u \cdot_{w,y} (v - v_1) = u \cdot_{w,y} v - u \cdot_{w,y} v_1$ and $(v - v_1) \cdot_{w,y} u = v \cdot_{w,y} u - v_1 \cdot_{w,y} u$ and $(v + v_1) \cdot_{w,y} u = v \cdot_{w,y} u + v_1 \cdot_{w,y} u$.

(33) Suppose w, y span the space. Let u, v be vectors of V and a be a real number. Then $(a \cdot u) \cdot_{w,y} v = a \cdot u \cdot_{w,y} v$ and $u \cdot_{w,y} (a \cdot v) = a \cdot u \cdot_{w,y} v$ and $(a \cdot u) \cdot_{w,y} v = u \cdot_{w,y} v \cdot a$ and $u \cdot_{w,y} (a \cdot v) = u \cdot_{w,y} v \cdot a$.

(34) Suppose w, y span the space. Let u, v be vectors of V. Then u, v are orthogonal w.r.t. w, y if and only if $u \cdot_{w,y} v = 0$.

(35) Suppose w, y span the space. Let u, v, u_1, v_1 be vectors of V. Then u, v, u_1 and v_1 are orthogonal w.r.t. w, y if and only if $(v-u) \cdot_{w,y} (v_1 - u_1) = 0$.

(36) If w, y span the space, then for all vectors u, v, v_1 of V holds $2 \cdot u \cdot_{w,y} (v \# v_1) = u \cdot_{w,y} v + u \cdot_{w,y} v_1$.

(37) If w, y span the space, then for all vectors u, v of V such that $u \neq v$ holds $(u-v) \cdot_{w,y} (u-v) \neq 0$.

(38) Suppose w, y span the space. Let p, q, u, v, v' be vectors of V and A be a real number. Suppose that

(i) p, q and u, v form a directed trapezium w.r.t. w, y,

(ii) $p \neq q$,

(iii) $A = ((p-q) \cdot_{w,y} (p+q) - 2 \cdot (p-q) \cdot_{w,y} u) \cdot (p-q) \cdot_{w,y} (p-q)^{-1}$, and

(iv) $v' = u + A \cdot (p-q)$.

Then $v = v'$.

(39) Suppose w, y span the space. Let u, u', u_1, u_2, v_1, v_2, t_1, t_2, w_1, w_2 be vectors of V. Suppose that

(i) $u \neq u'$,

(ii) u, u' and u_1, t_1 form a directed trapezium w.r.t. w, y,

(iii) u, u' and u_2, t_2 form a directed trapezium w.r.t. w, y,

(iv) u, u' and v_1, w_1 form a directed trapezium w.r.t. w, y,

(v) u, u' and v_2, w_2 form a directed trapezium w.r.t. w, y, and

(vi) $u_1, u_2 \upuparrows v_1, v_2$.

Then $t_1, t_2 \upuparrows w_1, w_2$.

(40) Suppose w, y span the space. Let u, u', u_1, u_2, v_1, t_1, t_2, w_1 be vectors of V. Suppose that

(i) $u \neq u'$,

(ii) u, u' and u_1, t_1 form a directed trapezium w.r.t. w, y,

(iii) u, u' and u_2, t_2 form a directed trapezium w.r.t. w, y,

(iv) u, u' and v_1, w_1 form a directed trapezium w.r.t. w, y, and

(v) $v_1 = u_1 \# u_2$.

Then $w_1 = t_1 \# t_2$.

(41) Suppose w, y span the space. Let u, u', u_1, u_2, t_1, t_2 be vectors of V. Suppose that

(i) $u \neq u'$,

(ii) u, u' and u_1, t_1 form a directed trapezium w.r.t. w, y, and

(iii) u, u' and u_2, t_2 form a directed trapezium w.r.t. w, y.

Then u, u' and $u_1 \# u_2$, $t_1 \# t_2$ form a directed trapezium w.r.t. w, y.

(42) Suppose w, y span the space. Let u, u', u_1, u_2, v_1, v_2, t_1, t_2, w_1, w_2 be vectors of V. Suppose that

(i) $u \neq u'$,

(ii) u, u' and u_1, t_1 form a directed trapezium w.r.t. w, y,

(iii) u, u' and u_2, t_2 form a directed trapezium w.r.t. w, y,

(iv) u, u' and v_1, w_1 form a directed trapezium w.r.t. w, y,

(v) u, u' and v_2, w_2 form a directed trapezium w.r.t. w, y, and

(vi) u_1, u_2, v_1 and v_2 are orthogonal w.r.t. w, y.

Then t_1, t_2, w_1 and w_2 are orthogonal w.r.t. w, y.

(43) Let u, u', u_1, u_2, v_1, v_2, t_1, t_2, w_1, w_2 be vectors of V. Suppose that w, y span the space and $u \neq u'$ and u, u' and u_1, t_1 form a directed trapezium w.r.t. w, y and u, u' and u_2, t_2 form a directed trapezium w.r.t. w, y and u, u' and v_1, w_1 form a directed trapezium w.r.t. w, y and u, u' and v_2, w_2 form a directed trapezium w.r.t. w, y and u_1, u_2 and v_1, v_2 form a directed trapezium w.r.t. w, y. Then t_1, t_2 and w_1, w_2 form a directed trapezium w.r.t. w, y.

Let us consider V and let us consider w, y. The directed trapezium relation defined over V in the basis w, y yielding a binary relation on $[\!:$the carrier of V, the carrier of $V\!:]$ is defined by the condition (Def. 7).

(Def. 7) The following statements are equivalent

(i) $\langle x, z\rangle \in$ the directed trapezium relation defined over V in the basis w, y,

(ii) there exist u, u_1, v, v_1 such that $x = \langle u, u_1\rangle$ and $z = \langle v, v_1\rangle$ and u, u_1 and v, v_1 form a directed trapezium w.r.t. w, y.

One can prove the following proposition

(44) The following statements are equivalent

(i) $\langle\langle u, v\rangle, \langle u_1, v_1\rangle\rangle \in$ the directed trapezium relation defined over V in the basis w, y,

(ii) u, v and u_1, v_1 form a directed trapezium w.r.t. w, y.

Let us consider V. The midpoint operation in V yields a binary operation on the carrier of V and is defined by:

(Def. 8) For all u, v holds (the midpoint operation in $V)(u, v) = u\#v$.

We consider affine midpoint structures as extensions of affine structure and midpoint algebra structure as systems

⟨ a carrier, a midpoint operation, a congruence ⟩,

where the carrier is a set, the midpoint operation is a binary operation on the carrier, and the congruence is a binary relation on $[\!:$the carrier, the carrier$\!:]$.

One can check that there exists an affine midpoint structure which is non empty and strict.

Let us consider V, w, y. The directed trapezium space defined over V in the basis w, y yields a strict affine midpoint structure and is defined by the condition (Def. 9).

(Def. 9) The directed trapezium space defined over V in the basis w, $y = \langle$the carrier of V, the midpoint operation in V, the directed trapezium relation defined over V in the basis w, $y\rangle$.

Let us consider V, w, y. One can verify that the directed trapezium space defined over V in the basis w, y is non empty.

Let A_1 be an affine midpoint structure. The affine reduct of A_1 yields a strict affine structure and is defined by:

(Def. 10) The affine reduct of $A_1 = \langle$the carrier of A_1, the congruence of $A_1\rangle$.

Let A_1 be a non empty affine midpoint structure. Note that the affine reduct of A_1 is non empty.

Let A_1 be a non empty affine midpoint structure and let a, b be elements of A_1. The functor $a\#b$ yields an element of A_1 and is defined by:

(Def. 12)[2] $a\#b =$ (the midpoint operation of $A_1)(a, b)$.

In the sequel a, b, a_1, b_1 denote elements of the directed trapezium space defined over V in the basis w, y.

Next we state three propositions:

[2] The definition (Def. 11) has been removed.

(46)[3] Let x be a set. Then x is an element of the carrier of (the directed trapezium space defined over V in the basis w, y) if and only if x is a vector of V.

(47) Suppose w, y span the space and $u = a$ and $v = b$ and $u_1 = a_1$ and $v_1 = b_1$. Then $a,b \upharpoonleft\!\upharpoonright a_1,b_1$ if and only if u, v and u_1, v_1 form a directed trapezium w.r.t. w, y.

(48) If w, y span the space and $u = a$ and $v = b$, then $u\#v = a\#b$.

Let I_1 be a non empty affine midpoint structure. We say that I_1 is ordered midpoint trapezium space-like if and only if the condition (Def. 13) is satisfied.

(Def. 13) Let a, b, c, d, a_1, b_1, c_1, d_1, p, q be elements of I_1. Then $a\#b = b\#a$ and $a\#a = a$ and $(a\#b)\#(c\#d) = a\#c\#(b\#d)$ and there exists an element p of I_1 such that $p\#a = b$ and if $a\#b = a\#c$, then $b = c$ and if $a,b \upharpoonleft\!\upharpoonright c,d$, then $a,b \upharpoonleft\!\upharpoonright a\#c,b\#d$ and if $a,b \upharpoonleft\!\upharpoonright c,d$, then $a,b \upharpoonleft\!\upharpoonright a\#d,b\#c$ or $a,b \upharpoonleft\!\upharpoonright b\#c,a\#d$ and if $a,b \upharpoonleft\!\upharpoonright c,d$ and $a\#a_1 = p$ and $b\#b_1 = p$ and $c\#c_1 = p$ and $d\#d_1 = p$, then $a_1,b_1 \upharpoonleft\!\upharpoonright c_1,d_1$ and if $p \neq q$ and $p,q \upharpoonleft\!\upharpoonright a,a_1$ and $p,q \upharpoonleft\!\upharpoonright b,b_1$ and $p,q \upharpoonleft\!\upharpoonright c,c_1$ and $p,q \upharpoonleft\!\upharpoonright d,d_1$ and $a,b \upharpoonleft\!\upharpoonright c,d$, then $a_1,b_1 \upharpoonleft\!\upharpoonright c_1,d_1$ and if $a,b \upharpoonleft\!\upharpoonright b,c$, then $a = b$ and $b = c$ and if $a,b \upharpoonleft\!\upharpoonright a_1,b_1$ and $a,b \upharpoonleft\!\upharpoonright c_1,d_1$ and $a \neq b$, then $a_1,b_1 \upharpoonleft\!\upharpoonright c_1,d_1$ and if $a,b \upharpoonleft\!\upharpoonright c,d$, then $c,d \upharpoonleft\!\upharpoonright a,b$ and $b,a \upharpoonleft\!\upharpoonright d,c$ and there exists an element d of I_1 such that $a,b \upharpoonleft\!\upharpoonright c,d$ or $a,b \upharpoonleft\!\upharpoonright d,c$ and if $a,b \upharpoonleft\!\upharpoonright c,p$ and $a,b \upharpoonleft\!\upharpoonright c,q$, then $a = b$ or $p = q$.

One can verify that there exists a non empty affine midpoint structure which is strict and ordered midpoint trapezium space-like.

An ordered midpoint trapezium space is an ordered midpoint trapezium space-like non empty affine midpoint structure.

Next we state the proposition

(49) Suppose w, y span the space. Then the directed trapezium space defined over V in the basis w, y is an ordered midpoint trapezium space.

Let I_1 be a non empty affine structure. We say that I_1 is ordered trapezium space-like if and only if the condition (Def. 14) is satisfied.

(Def. 14) Let a, b, c, d, a_1, b_1, c_1, d_1, p, q be elements of I_1. Then

(i) if $a,b \upharpoonleft\!\upharpoonright b,c$, then $a = b$ and $b = c$,

(ii) if $a,b \upharpoonleft\!\upharpoonright a_1,b_1$ and $a,b \upharpoonleft\!\upharpoonright c_1,d_1$ and $a \neq b$, then $a_1,b_1 \upharpoonleft\!\upharpoonright c_1,d_1$,

(iii) if $a,b \upharpoonleft\!\upharpoonright c,d$, then $c,d \upharpoonleft\!\upharpoonright a,b$ and $b,a \upharpoonleft\!\upharpoonright d,c$,

(iv) there exists an element d of I_1 such that $a,b \upharpoonleft\!\upharpoonright c,d$ or $a,b \upharpoonleft\!\upharpoonright d,c$, and

(v) if $a,b \upharpoonleft\!\upharpoonright c,p$ and $a,b \upharpoonleft\!\upharpoonright c,q$, then $a = b$ or $p = q$.

One can verify that there exists a non empty affine structure which is strict and ordered trapezium space-like.

An ordered trapezium space is an ordered trapezium space-like non empty affine structure.

Let M_1 be an ordered midpoint trapezium space. One can check that the affine reduct of M_1 is ordered trapezium space-like.

We use the following convention: O_1 denotes an ordered trapezium space, a, b, c, d denote elements of O_1, and a', b', c', d' denote elements of $\Lambda(O_1)$.

Next we state two propositions:

(50) For every set x holds x is an element of O_1 iff x is an element of $\Lambda(O_1)$.

(51) If $a = a'$ and $b = b'$ and $c = c'$ and $d = d'$, then $a',b' \upharpoonleft\!\upharpoonright c',d'$ iff $a,b \upharpoonleft\!\upharpoonright c,d$ or $a,b \upharpoonleft\!\upharpoonright d,c$.

Let I_1 be a non empty affine structure. We say that I_1 is trapezium space-like if and only if the condition (Def. 15) is satisfied.

[3] The proposition (45) has been removed.

(Def. 15) Let a', b', c', d', p', q' be elements of I_1. Then

(i) $a',b' \upharpoonleft\upharpoonright b',a'$,

(ii) if $a',b' \upharpoonleft\upharpoonright c',d'$ and $a',b' \upharpoonleft\upharpoonright c',q'$, then $a' = b'$ or $d' = q'$,

(iii) if $p' \neq q'$ and $p',q' \upharpoonleft\upharpoonright a',b'$ and $p',q' \upharpoonleft\upharpoonright c',d'$, then $a',b' \upharpoonleft\upharpoonright c',d'$,

(iv) if $a',b' \upharpoonleft\upharpoonright c',d'$, then $c',d' \upharpoonleft\upharpoonright a',b'$, and

(v) there exists an element x' of I_1 such that $a',b' \upharpoonleft\upharpoonright c',x'$.

Let us note that there exists a non empty affine structure which is strict and trapezium space-like.

A trapezium space is a trapezium space-like non empty affine structure.

Let O_1 be an ordered trapezium space. One can verify that $\Lambda(O_1)$ is trapezium space-like.

Let I_1 be a non empty affine structure. We say that I_1 is regular if and only if the condition (Def. 16) is satisfied.

(Def. 16) Let p, q, a, a_1, b, b_1, c, c_1, d, d_1 be elements of I_1. If $p \neq q$ and $p,q \upharpoonleft\upharpoonright a,a_1$ and $p,q \upharpoonleft\upharpoonright b,b_1$ and $p,q \upharpoonleft\upharpoonright c,c_1$ and $p,q \upharpoonleft\upharpoonright d,d_1$ and $a,b \upharpoonleft\upharpoonright c,d$, then $a_1,b_1 \upharpoonleft\upharpoonright c_1,d_1$.

Let us observe that there exists a non empty ordered trapezium space which is strict and regular.

Let M_1 be an ordered midpoint trapezium space. Observe that the affine reduct of M_1 is regular.

REFERENCES

[1] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/binop_1.html`.

[2] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[3] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[4] Michał Muzalewski. Midpoint algebras. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/midsp_1.html`.

[5] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical metric affine spaces and planes. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/analmetr.html`.

[6] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical ordered affine spaces. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/analoaf.html`.

[7] Henryk Oryszczyszyn and Krzysztof Prażmowski. Ordered affine spaces defined in terms of directed parallelity — part I. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/diraf.html`.

[8] Andrzej Trybulec. Domains and their Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/domain_1.html`.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[10] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/rlvect_1.html`.

[11] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[12] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

Received October 29, 1990

Published January 2, 2004