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Logic Gates and Logical Equivalence of Adders

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Summary. This is an experimental article which shows that logical correctness of logic circuits can be easily proven by the Mizar system. First, we define the notion of logic gates. Then we prove that an MSB carry of '4 Bit Carry Skip Adder' is equivalent to an MSB carry of a normal 4 bit adder. In the last theorem, we show that outputs of the '4 Bit Carry Look Ahead Adder' are equivalent to the corresponding outputs of the normal 4 bits adder. The policy here is as follows: when the functional (semantic) correctness of a system is already proven, and the correspondence of the system to a (normal) logic circuit is given, it is enough to prove the correctness of the new circuit if we only prove the logical equivalence between them. Although the article is very fundamental (it contains few environment files), it can be applied to real problems. The key of the method introduced here is to put the specification of the logic circuit into the Mizar propositional formulae, and to use the strong inference ability of the Mizar checker. The proof is done formally so that the automation of the proof writing is possible. Even in the 5.3.07 version of Mizar, it can handle a formulae of more than 100 lines, and a formula which contains more than 100 variables. This means that the Mizar system is enough to prove logical correctness of middle scaled logic circuits.

MML Identifier: `GATE_1`.

WWW: `http://mizar.org/JFM/Vol11/gate_1.html`

1. Definition of Logical Values and Logic Gates

Let a be a set. We introduce NE a as an antonym of a is empty.

Next we state two propositions:

(1) For every set a such that $a = \{\emptyset\}$ holds NE a.

(2) There exists a set a such that NE a.

Let a be a set. The functor NOT1 a is defined as follows:

(Def. 1) $\mathrm{NOT1}\,a = \begin{cases} \emptyset, \text{ if NE } a, \\ \{\emptyset\}, \text{ otherwise.} \end{cases}$

We now state the proposition

(4)[1] For every set a holds NE NOT1 a iff not NE a.

In the sequel a, b, c, d, e, f, g, h are sets.

One can prove the following proposition

(5) NE NOT1 $\emptyset$.

[1] The proposition (3) has been removed.

Let a, b be sets. The functor $\mathrm{AND2}(a,b)$ is defined as follows:

(Def. 2) $\mathrm{AND2}(a,b) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ and NE } b, \\ \emptyset, & \text{otherwise.} \end{cases}$

Next we state the proposition

(6) NE $\mathrm{AND2}(a,b)$ iff NE a and NE b.

Let a, b be sets. The functor $\mathrm{OR2}(a,b)$ is defined as follows:

(Def. 3) $\mathrm{OR2}(a,b) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ or NE } b, \\ \emptyset, & \text{otherwise.} \end{cases}$

Next we state the proposition

(7) NE $\mathrm{OR2}(a,b)$ iff NE a or NE b.

Let a, b be sets. The functor $\mathrm{XOR2}(a,b)$ is defined as follows:

(Def. 4) $\mathrm{XOR2}(a,b) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ and not NE } b \text{ or not NE } a \text{ and NE } b, \\ \emptyset, & \text{otherwise.} \end{cases}$

One can prove the following propositions:

(8) NE $\mathrm{XOR2}(a,b)$ iff NE a and not NE b or not NE a and NE b.

(9) NE $\mathrm{XOR2}(a,a)$ iff *contradiction*.[2]

(10) NE $\mathrm{XOR2}(a,\emptyset)$ iff NE a.

(11) NE $\mathrm{XOR2}(a,b)$ iff NE $\mathrm{XOR2}(b,a)$.

Let a, b be sets. The functor $\mathrm{EQV2}(a,b)$ is defined by:

(Def. 5) $\mathrm{EQV2}(a,b) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ iff NE } b, \\ \emptyset, & \text{otherwise.} \end{cases}$

The following two propositions are true:

(12) NE $\mathrm{EQV2}(a,b)$ iff NE a iff NE b.

(13) NE $\mathrm{EQV2}(a,b)$ iff not NE $\mathrm{XOR2}(a,b)$.

Let a, b be sets. The functor $\mathrm{NAND2}(a,b)$ is defined as follows:

(Def. 6) $\mathrm{NAND2}(a,b) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ or not NE } b, \\ \emptyset, & \text{otherwise.} \end{cases}$

One can prove the following proposition

(14) NE $\mathrm{NAND2}(a,b)$ iff not NE a or not NE b.

Let a, b be sets. The functor $\mathrm{NOR2}(a,b)$ is defined by:

(Def. 7) $\mathrm{NOR2}(a,b) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ and not NE } b, \\ \emptyset, & \text{otherwise.} \end{cases}$

The following proposition is true

(15) NE $\mathrm{NOR2}(a,b)$ iff not NE a and not NE b.

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[2] This definition is absolutely permissive, i.e. we assume a *contradiction*, but we are interested only in the type of the functor 'choose'.

Let a, b, c be sets. The functor $\mathrm{AND3}(a,b,c)$ is defined by:

(Def. 8) $\mathrm{AND3}(a,b,c) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ and NE } b \text{ and NE } c, \\ \emptyset, & \text{otherwise.} \end{cases}$

The following proposition is true

(16) NE $\mathrm{AND3}(a,b,c)$ iff NE a and NE b and NE c.

Let a, b, c be sets. The functor $\mathrm{OR3}(a,b,c)$ is defined as follows:

(Def. 9) $\mathrm{OR3}(a,b,c) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ or NE } b \text{ or NE } c, \\ \emptyset, & \text{otherwise.} \end{cases}$

One can prove the following proposition

(17) NE $\mathrm{OR3}(a,b,c)$ iff NE a or NE b or NE c.

Let a, b, c be sets. The functor $\mathrm{XOR3}(a,b,c)$ is defined by:

(Def. 10) $\mathrm{XOR3}(a,b,c) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ and not NE } b \text{ or not NE } a \text{ and NE } b \text{ but not NE } c \text{ or not NE } a \text{ or not NE } b \text{ but not NE } a \text{ or not NE } b \text{ and NE } c, \\ \emptyset, & \text{otherwise.} \end{cases}$

One can prove the following proposition

(18) NE $\mathrm{XOR3}(a,b,c)$ if and only if one of the following conditions is satisfied:

(i) NE a and not NE b or not NE a and NE b but not NE c, or

(ii) not NE a or not NE b but not NE a or not NE b and NE c.

Let a, b, c be sets. The functor $\mathrm{MAJ3}(a,b,c)$ is defined as follows:

(Def. 11) $\mathrm{MAJ3}(a,b,c) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ and NE } b \text{ or NE } b \text{ and NE } c \text{ or NE } c \text{ and NE } a, \\ \emptyset, & \text{otherwise.} \end{cases}$

We now state the proposition

(19) NE $\mathrm{MAJ3}(a,b,c)$ iff NE a and NE b or NE b and NE c or NE c and NE a.

Let a, b, c be sets. The functor $\mathrm{NAND3}(a,b,c)$ is defined as follows:

(Def. 12) $\mathrm{NAND3}(a,b,c) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ or not NE } b \text{ or not NE } c, \\ \emptyset, & \text{otherwise.} \end{cases}$

Next we state the proposition

(20) NE $\mathrm{NAND3}(a,b,c)$ iff not NE a or not NE b or not NE c.

Let a, b, c be sets. The functor $\mathrm{NOR3}(a,b,c)$ is defined by:

(Def. 13) $\mathrm{NOR3}(a,b,c) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ and not NE } b \text{ and not NE } c, \\ \emptyset, & \text{otherwise.} \end{cases}$

One can prove the following proposition

(21) NE $\mathrm{NOR3}(a,b,c)$ iff not NE a and not NE b and not NE c.

Let a, b, c, d be sets. The functor $\mathrm{AND4}(a,b,c,d)$ is defined by:

(Def. 14) $\mathrm{AND4}(a,b,c,d) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ and NE } b \text{ and NE } c \text{ and NE } d, \\ \emptyset, & \text{otherwise.} \end{cases}$

One can prove the following proposition

(22) NE $\mathrm{AND4}(a,b,c,d)$ iff NE a and NE b and NE c and NE d.

Let a, b, c, d be sets. The functor $\mathrm{OR4}(a,b,c,d)$ is defined by:

(Def. 15) $\mathrm{OR4}(a,b,c,d) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ or NE } b \text{ or NE } c \text{ or NE } d, \\ \emptyset, & \text{otherwise.} \end{cases}$

Next we state the proposition

(23) NE $\mathrm{OR4}(a,b,c,d)$ iff NE a or NE b or NE c or NE d.

Let a, b, c, d be sets. The functor $\mathrm{NAND4}(a,b,c,d)$ is defined as follows:

(Def. 16) $\mathrm{NAND4}(a,b,c,d) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ or not NE } b \text{ or not NE } c \text{ or not NE } d, \\ \emptyset, & \text{otherwise.} \end{cases}$

The following proposition is true

(24) NE $\mathrm{NAND4}(a,b,c,d)$ iff not NE a or not NE b or not NE c or not NE d.

Let a, b, c, d be sets. The functor $\mathrm{NOR4}(a,b,c,d)$ is defined as follows:

(Def. 17) $\mathrm{NOR4}(a,b,c,d) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ and not NE } b \text{ and not NE } c \text{ and not NE } d, \\ \emptyset, & \text{otherwise.} \end{cases}$

One can prove the following proposition

(25) NE $\mathrm{NOR4}(a,b,c,d)$ iff not NE a and not NE b and not NE c and not NE d.

Let a, b, c, d, e be sets. The functor $\mathrm{AND5}(a,b,c,d,e)$ is defined as follows:

(Def. 18) $\mathrm{AND5}(a,b,c,d,e) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ and NE } b \text{ and NE } c \text{ and NE } d \text{ and NE } e, \\ \emptyset, & \text{otherwise.} \end{cases}$

We now state the proposition

(26) NE $\mathrm{AND5}(a,b,c,d,e)$ iff NE a and NE b and NE c and NE d and NE e.

Let a, b, c, d, e be sets. The functor $\mathrm{OR5}(a,b,c,d,e)$ is defined by:

(Def. 19) $\mathrm{OR5}(a,b,c,d,e) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ or NE } b \text{ or NE } c \text{ or NE } d \text{ or NE } e, \\ \emptyset, & \text{otherwise.} \end{cases}$

The following proposition is true

(27) NE $\mathrm{OR5}(a,b,c,d,e)$ iff NE a or NE b or NE c or NE d or NE e.

Let a, b, c, d, e be sets. The functor $\mathrm{NAND5}(a,b,c,d,e)$ is defined by:

(Def. 20) $\mathrm{NAND5}(a,b,c,d,e) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ or not NE } b \text{ or not NE } c \text{ or not NE } d \text{ or not NE } e, \\ \emptyset, & \text{otherwise.} \end{cases}$

The following proposition is true

(28) NE $\mathrm{NAND5}(a,b,c,d,e)$ iff not NE a or not NE b or not NE c or not NE d or not NE e.

Let a, b, c, d, e be sets. The functor $\mathrm{NOR5}(a,b,c,d,e)$ is defined as follows:

(Def. 21) $\mathrm{NOR5}(a,b,c,d,e) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ and not NE } b \text{ and not NE } c \text{ and not NE } d \text{ and not NE } e, \\ \emptyset, & \text{otherwise.} \end{cases}$

The following proposition is true

(29) NE $\mathrm{NOR5}(a,b,c,d,e)$ iff not NE a and not NE b and not NE c and not NE d and not NE e.

Let a, b, c, d, e, f be sets. The functor $\mathrm{AND6}(a,b,c,d,e,f)$ is defined by:

(Def. 22) $\mathrm{AND6}(a,b,c,d,e,f) = \begin{cases} \mathrm{NOT1}\,\emptyset, \text{ if NE } a \text{ and NE } b \text{ and NE } c \text{ and NE } d \text{ and NE } e \text{ and NE } f, \\ \emptyset, \text{ otherwise.} \end{cases}$

One can prove the following proposition

(30) NE $\mathrm{AND6}(a,b,c,d,e,f)$ iff NE a and NE b and NE c and NE d and NE e and NE f.

Let a, b, c, d, e, f be sets. The functor $\mathrm{OR6}(a,b,c,d,e,f)$ is defined as follows:

(Def. 23) $\mathrm{OR6}(a,b,c,d,e,f) = \begin{cases} \mathrm{NOT1}\,\emptyset, \text{ if NE } a \text{ or NE } b \text{ or NE } c \text{ or NE } d \text{ or NE } e \text{ or NE } f, \\ \emptyset, \text{ otherwise.} \end{cases}$

We now state the proposition

(31) NE $\mathrm{OR6}(a,b,c,d,e,f)$ iff NE a or NE b or NE c or NE d or NE e or NE f.

Let a, b, c, d, e, f be sets. The functor $\mathrm{NAND6}(a,b,c,d,e,f)$ is defined as follows:

(Def. 24) $\mathrm{NAND6}(a,b,c,d,e,f) = \begin{cases} \mathrm{NOT1}\,\emptyset, \text{ if not NE } a \text{ or not NE } b \text{ or not NE } c \text{ or not NE } d \text{ or not NE } e \text{ or not NE } f, \\ \emptyset, \text{ otherwise.} \end{cases}$

Next we state the proposition

(32) NE $\mathrm{NAND6}(a,b,c,d,e,f)$ iff not NE a or not NE b or not NE c or not NE d or not NE e or not NE f.

Let a, b, c, d, e, f be sets. The functor $\mathrm{NOR6}(a,b,c,d,e,f)$ is defined as follows:

(Def. 25) $\mathrm{NOR6}(a,b,c,d,e,f) = \begin{cases} \mathrm{NOT1}\,\emptyset, \text{ if not NE } a \text{ and not NE } b \text{ and not NE } c \text{ and not NE } d \text{ and not NE } e \text{ and not NE } f, \\ \emptyset, \text{ otherwise.} \end{cases}$

Next we state the proposition

(33) NE $\mathrm{NOR6}(a,b,c,d,e,f)$ iff not NE a and not NE b and not NE c and not NE d and not NE e and not NE f.

Let a, b, c, d, e, f, g be sets. The functor $\mathrm{AND7}(a,b,c,d,e,f,g)$ is defined as follows:

(Def. 26) $\mathrm{AND7}(a,b,c,d,e,f,g) = \begin{cases} \mathrm{NOT1}\,\emptyset, \text{ if NE } a \text{ and NE } b \text{ and NE } c \text{ and NE } d \text{ and NE } e \text{ and NE } f \text{ and NE } g, \\ \emptyset, \text{ otherwise.} \end{cases}$

One can prove the following proposition

(34) NE $\mathrm{AND7}(a,b,c,d,e,f,g)$ iff NE a and NE b and NE c and NE d and NE e and NE f and NE g.

Let a, b, c, d, e, f, g be sets. The functor $\mathrm{OR7}(a,b,c,d,e,f,g)$ is defined by:

(Def. 27) $\mathrm{OR7}(a,b,c,d,e,f,g) = \begin{cases} \mathrm{NOT1}\,\emptyset, \text{ if NE } a \text{ or NE } b \text{ or NE } c \text{ or NE } d \text{ or NE } e \text{ or NE } f \text{ or NE } g, \\ \emptyset, \text{ otherwise.} \end{cases}$

One can prove the following proposition

(35) NE $\mathrm{OR7}(a,b,c,d,e,f,g)$ iff NE a or NE b or NE c or NE d or NE e or NE f or NE g.

Let a, b, c, d, e, f, g be sets. The functor $\mathrm{NAND7}(a,b,c,d,e,f,g)$ is defined as follows:

(Def. 28) $\mathrm{NAND7}(a,b,c,d,e,f,g) = \begin{cases} \mathrm{NOT1}\,\emptyset, \text{ if not NE } a \text{ or not NE } b \text{ or not NE } c \text{ or not NE } d \text{ or not NE } e \text{ or not NE } f \text{ or not NE } g, \\ \emptyset, \text{ otherwise.} \end{cases}$

Next we state the proposition

(36) NE $\mathrm{NAND7}(a,b,c,d,e,f,g)$ iff not NE a or not NE b or not NE c or not NE d or not NE e or not NE f or not NE g.

Let a, b, c, d, e, f, g be sets. The functor $\mathrm{NOR7}(a,b,c,d,e,f,g)$ is defined by:

(Def. 29) $$\mathrm{NOR7}(a,b,c,d,e,f,g) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ and not NE } b \text{ and not NE } c \text{ and not NE } d \text{ and not NE } e \text{ and not NE } f \text{ and not NE } g, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We now state the proposition

(37) NE $\mathrm{NOR7}(a,b,c,d,e,f,g)$ iff not NE a and not NE b and not NE c and not NE d and not NE e and not NE f and not NE g.

Let a, b, c, d, e, f, g, h be sets. The functor $\mathrm{AND8}(a,b,c,d,e,f,g,h)$ is defined as follows:

(Def. 30) $$\mathrm{AND8}(a,b,c,d,e,f,g,h) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ and NE } b \text{ and NE } c \text{ and NE } d \text{ and NE } e \text{ and NE } f \text{ and NE } g \text{ and NE } h, \\ \emptyset, & \text{otherwise.} \end{cases}$$

One can prove the following proposition

(38) NE $\mathrm{AND8}(a,b,c,d,e,f,g,h)$ iff NE a and NE b and NE c and NE d and NE e and NE f and NE g and NE h.

Let a, b, c, d, e, f, g, h be sets. The functor $\mathrm{OR8}(a,b,c,d,e,f,g,h)$ is defined as follows:

(Def. 31) $$\mathrm{OR8}(a,b,c,d,e,f,g,h) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ or NE } b \text{ or NE } c \text{ or NE } d \text{ or NE } e \text{ or NE } f \text{ or NE } g \text{ or NE } h, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The following proposition is true

(39) NE $\mathrm{OR8}(a,b,c,d,e,f,g,h)$ iff NE a or NE b or NE c or NE d or NE e or NE f or NE g or NE h.

Let a, b, c, d, e, f, g, h be sets. The functor $\mathrm{NAND8}(a,b,c,d,e,f,g,h)$ is defined as follows:

(Def. 32) $$\mathrm{NAND8}(a,b,c,d,e,f,g,h) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ or not NE } b \text{ or not NE } c \text{ or not NE } d \text{ or not NE } e \text{ or not NE } f \text{ or not NE } g \text{ or not NE } h, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Next we state the proposition

(40) NE $\mathrm{NAND8}(a,b,c,d,e,f,g,h)$ iff not NE a or not NE b or not NE c or not NE d or not NE e or not NE f or not NE g or not NE h.

Let a, b, c, d, e, f, g, h be sets. The functor $\mathrm{NOR8}(a,b,c,d,e,f,g,h)$ is defined as follows:

(Def. 33) $$\mathrm{NOR8}(a,b,c,d,e,f,g,h) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if not NE } a \text{ and not NE } b \text{ and not NE } c \text{ and not NE } d \text{ and not NE } e \text{ and not NE } f \text{ and not NE } g \text{ and not NE } h, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We now state the proposition

(41) NE $\mathrm{NOR8}(a,b,c,d,e,f,g,h)$ iff not NE a and not NE b and not NE c and not NE d and not NE e and not NE f and not NE g and not NE h.

2. LOGICAL EQUIVALENCE OF 4 BITS ADDERS

The following proposition is true

(42) Let c_1, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, c_2, c_3, c_4, c_5, n_1, n_2, n_3, n_4, n, c_6 be sets. Suppose that NE c_2 iff NE $\mathrm{MAJ3}(x_1,y_1,c_1)$ and NE c_3 iff NE $\mathrm{MAJ3}(x_2,y_2,c_2)$ and NE c_4 iff NE $\mathrm{MAJ3}(x_3,y_3,c_3)$ and NE c_5 iff NE $\mathrm{MAJ3}(x_4,y_4,c_4)$ and NE n_1 iff NE $\mathrm{OR2}(x_1,y_1)$ and NE n_2 iff NE $\mathrm{OR2}(x_2,y_2)$ and NE n_3 iff NE $\mathrm{OR2}(x_3,y_3)$ and NE n_4 iff NE $\mathrm{OR2}(x_4,y_4)$ and NE n iff NE $\mathrm{AND5}(c_1,n_1,n_2,n_3,n_4)$ and NE c_6 iff NE $\mathrm{OR2}(c_5,n)$. Then NE c_5 if and only if NE c_6.

Let a, b be sets. The functor $\mathrm{MODADD2}(a,b)$ is defined as follows:

(Def. 34) $\mathrm{MODADD2}(a,b) = \begin{cases} \mathrm{NOT1}\,\emptyset, & \text{if NE } a \text{ or NE } b \text{ but NE } a \text{ but NE } b, \\ \emptyset, & \text{otherwise.} \end{cases}$

We now state the proposition

(43) NE MODADD2(a,b) iff NE a or NE b but NE a but NE b.

Let a, b, c be sets. We introduce ADD1(a,b,c) as a synonym of XOR3(a,b,c). We introduce CARR1(a,b,c) as a synonym of MAJ3(a,b,c).

Let a_1, b_1, a_2, b_2, c be sets. The functor ADD2(a_2,b_2,a_1,b_1,c) is defined by:

(Def. 37)[3] ADD2$(a_2,b_2,a_1,b_1,c) =$ XOR3$(a_2,b_2,$CARR1$(a_1,b_1,c))$.

Let a_1, b_1, a_2, b_2, c be sets. The functor CARR2(a_2,b_2,a_1,b_1,c) is defined as follows:

(Def. 38) CARR2$(a_2,b_2,a_1,b_1,c) =$ MAJ3$(a_2,b_2,$CARR1$(a_1,b_1,c))$.

Let a_1, b_1, a_2, b_2, a_3, b_3, c be sets. The functor ADD3$(a_3,b_3,a_2,b_2,a_1,b_1,c)$ is defined by:

(Def. 39) ADD3$(a_3,b_3,a_2,b_2,a_1,b_1,c) =$ XOR3$(a_3,b_3,$CARR2$(a_2,b_2,a_1,b_1,c))$.

Let a_1, b_1, a_2, b_2, a_3, b_3, c be sets. The functor CARR3$(a_3,b_3,a_2,b_2,a_1,b_1,c)$ is defined by:

(Def. 40) CARR3$(a_3,b_3,a_2,b_2,a_1,b_1,c) =$ MAJ3$(a_3,b_3,$CARR2$(a_2,b_2,a_1,b_1,c))$.

Let a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, c be sets. The functor ADD4$(a_4,b_4,a_3,b_3,a_2,b_2,a_1,b_1,c)$ is defined by:

(Def. 41) ADD4$(a_4,b_4,a_3,b_3,a_2,b_2,a_1,b_1,c) =$ XOR3$(a_4,b_4,$CARR3$(a_3,b_3,a_2,b_2,a_1,b_1,c))$.

Let a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, c be sets. The functor CARR4$(a_4,b_4,a_3,b_3,a_2,b_2,a_1,b_1,c)$ is defined by:

(Def. 42) CARR4$(a_4,b_4,a_3,b_3,a_2,b_2,a_1,b_1,c) =$ MAJ3$(a_4,b_4,$CARR3$(a_3,b_3,a_2,b_2,a_1,b_1,c))$.

One can prove the following proposition

(44) Let c_1, x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, c_4, q_1, p_1, s_1, q_2, p_2, s_2, q_3, p_3, s_3, q_4, p_4, s_4, c_7, c_8, l_2, t_2, l_3, m_3, t_3, l_4, m_4, n_4, t_4, l_5, m_5, n_5, o_5, s_5, s_6, s_7, s_8 be sets such that NE q_1 iff NE NOR2(x_1,y_1) and NE p_1 iff NE NAND2(x_1,y_1) and NE s_1 iff NE MODADD2(x_1,y_1) and NE q_2 iff NE NOR2(x_2,y_2) and NE p_2 iff NE NAND2(x_2,y_2) and NE s_2 iff NE MODADD2(x_2,y_2) and NE q_3 iff NE NOR2(x_3,y_3) and NE p_3 iff NE NAND2(x_3,y_3) and NE s_3 iff NE MODADD2(x_3,y_3) and NE q_4 iff NE NOR2(x_4,y_4) and NE p_4 iff NE NAND2(x_4,y_4) and NE s_4 iff NE MODADD2(x_4,y_4) and NE c_7 iff NE NOT1 c_1 and NE c_8 iff NE NOT1 c_7 and NE s_5 iff NE XOR2(c_8,s_1) and NE l_2 iff NE AND2(c_7,p_1) and NE t_2 iff NE NOR2(l_2,q_1) and NE s_6 iff NE XOR2(t_2,s_2) and NE l_3 iff NE AND2(q_1,p_2) and NE m_3 iff NE AND3(p_2,p_1,c_7) and NE t_3 iff NE NOR3(l_3,m_3,q_2) and NE s_7 iff NE XOR2(t_3,s_3) and NE l_4 iff NE AND2(q_2,p_3) and NE m_4 iff NE AND3(q_1,p_3,p_2) and NE n_4 iff NE AND4(p_3,p_2,p_1,c_7) and NE t_4 iff NE NOR4(l_4,m_4,n_4,q_3) and NE s_8 iff NE XOR2(t_4,s_4) and NE l_5 iff NE AND2(q_3,p_4) and NE m_5 iff NE AND3(q_2,p_4,p_3) and NE n_5 iff NE AND4(q_1,p_4,p_3,p_2) and NE o_5 iff NE AND5(p_4,p_3,p_2,p_1,c_7) and NE c_4 iff NE NOR5(q_4,l_5,m_5,n_5,o_5). Then

(i) NE s_5 iff NE ADD1(x_1,y_1,c_1),

(ii) NE s_6 iff NE ADD2(x_2,y_2,x_1,y_1,c_1),

(iii) NE s_7 iff NE ADD3$(x_3,y_3,x_2,y_2,x_1,y_1,c_1)$,

(iv) NE s_8 iff NE ADD4$(x_4,y_4,x_3,y_3,x_2,y_2,x_1,y_1,c_1)$, and

(v) NE c_4 iff NE CARR4$(x_4,y_4,x_3,y_3,x_2,y_2,x_1,y_1,c_1)$.

[3] The definitions (Def. 35) and (Def. 36) have been removed.

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