JOURNAL OF FORMALIZED MATHEMATICS
Volume 4, Released 1992, Published 2003
Inst. of Computer Science, Univ. of Białystok

Sum and Product of Finite Sequences of Elements of a Field

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Summary. This article is concerned with a generalization of concepts introduced in [11], i.e., there are introduced the sum and the product of finite number of elements of any field. Moreover, the product of vectors which yields a vector is introduced. According to [11], some operations on i-tuples of elements of field are introduced: addition, subtraction, and complement. Some properties of the sum and the product of finite number of elements of a field are present.

MML Identifier: FVSUM_1.

WWW: http://mizar.org/JFM/Vol4/fvsum_1.html

The articles [18], [22], [19], [2], [23], [5], [7], [6], [3], [4], [16], [21], [17], [9], [8], [10], [15], [14], [1], [12], [20], and [13] provide the notation and terminology for this paper.

1. Auxiliary theorems

In this paper i, j, k denote natural numbers.

One can prove the following propositions:

(2)[1] For every Abelian non empty loop structure K holds the addition of K is commutative.

(3) For every add-associative non empty loop structure K holds the addition of K is associative.

(4) For every commutative non empty groupoid K holds the multiplication of K is commutative.

(6)[2] Let K be a commutative left unital non empty double loop structure. Then $\mathbf{1}_K$ is a unity w.r.t. the multiplication of K.

(7) For every commutative left unital non empty double loop structure K holds $\mathbf{1}_{\text{the multiplication of } K} = \mathbf{1}_K$.

(8) For every left zeroed right zeroed non empty loop structure K holds 0_K is a unity w.r.t. the addition of K.

(9) For every left zeroed right zeroed non empty loop structure K holds $\mathbf{1}_{\text{the addition of } K} = 0_K$.

(10) For every left zeroed right zeroed non empty loop structure K holds the addition of K has a unity.

[1] The proposition (1) has been removed.
[2] The proposition (5) has been removed.

(11) For every commutative left unital non empty double loop structure K holds the multiplication of K has a unity.

(12) Let K be a distributive non empty double loop structure. Then the multiplication of K is distributive w.r.t. the addition of K.

Let K be a non empty groupoid and let a be an element of K. The functor $\cdot^a$ yielding a unary operation on the carrier of K is defined by:

(Def. 1) $\cdot^a = (\text{the multiplication of } K)^{\circ}(a, \mathrm{id}_{\text{the carrier of } K})$.

Let K be a non empty loop structure. The functor $-_K$ yielding a binary operation on the carrier of K is defined as follows:

(Def. 2) $-_K = (\text{the addition of } K) \circ (\mathrm{id}_{\text{the carrier of } K}, \mathrm{comp}\, K)$.

We now state several propositions:

(14)[3] For every non empty loop structure K and for all elements a_1, a_2 of K holds $-_K(a_1, a_2) = a_1 - a_2$.

(15) Let K be a distributive non empty double loop structure and a be an element of K. Then $\cdot^a$ is distributive w.r.t. the addition of K.

(16) Let K be a left zeroed right zeroed add-associative right complementable non empty loop structure. Then $\mathrm{comp}\, K$ is an inverse operation w.r.t. the addition of K.

(17) Let K be a left zeroed right zeroed add-associative right complementable non empty loop structure. Then the addition of K has an inverse operation.

(18) Let K be a left zeroed right zeroed add-associative right complementable non empty loop structure. Then the inverse operation w.r.t. the addition of $K = \mathrm{comp}\, K$.

(19) Let K be a right zeroed add-associative right complementable Abelian non empty loop structure. Then $\mathrm{comp}\, K$ is distributive w.r.t. the addition of K.

2. SOME OPERATIONS ON i-TUPLES

Let K be a non empty loop structure and let p_1, p_2 be finite sequences of elements of the carrier of K. The functor $p_1 + p_2$ yielding a finite sequence of elements of the carrier of K is defined by:

(Def. 3) $p_1 + p_2 = (\text{the addition of } K)^{\circ}(p_1, p_2)$.

One can prove the following proposition

(21)[4] Let K be a non empty loop structure, p_1, p_2 be finite sequences of elements of the carrier of K, a_1, a_2 be elements of K, and i be a natural number. If $i \in \mathrm{dom}(p_1 + p_2)$ and $a_1 = p_1(i)$ and $a_2 = p_2(i)$, then $(p_1 + p_2)(i) = a_1 + a_2$.

Let us consider i, let K be a non empty loop structure, and let R_1, R_2 be elements of (the carrier of $K)^i$. Then $R_1 + R_2$ is an element of (the carrier of $K)^i$.

We now state several propositions:

(22) Let K be a non empty loop structure, a_1, a_2 be elements of K, and R_1, R_2 be elements of (the carrier of $K)^i$. If $j \in \mathrm{Seg}\, i$ and $a_1 = R_1(j)$ and $a_2 = R_2(j)$, then $(R_1 + R_2)(j) = a_1 + a_2$.

(23) Let K be a non empty loop structure and p be a finite sequence of elements of the carrier of K. Then $\varepsilon_{(\text{the carrier of } K)} + p = \varepsilon_{(\text{the carrier of } K)}$ and $p + \varepsilon_{(\text{the carrier of } K)} = \varepsilon_{(\text{the carrier of } K)}$.

[3] The proposition (13) has been removed.
[4] The proposition (20) has been removed.

(24) For every non empty loop structure K and for all elements a_1, a_2 of K holds $\langle a_1\rangle + \langle a_2\rangle = \langle a_1 + a_2\rangle$.

(25) For every non empty loop structure K and for all elements a_1, a_2 of K holds $i \mapsto a_1 + i \mapsto a_2 = i \mapsto (a_1 + a_2)$.

(26) For every Abelian non empty loop structure K and for all elements R_1, R_2 of (the carrier of $K)^i$ holds $R_1 + R_2 = R_2 + R_1$.

(27) Let K be an add-associative non empty loop structure and R_1, R_2, R_3 be elements of (the carrier of $K)^i$. Then $R_1 + (R_2 + R_3) = (R_1 + R_2) + R_3$.

(28) Let K be an Abelian left zeroed right zeroed non empty loop structure and R be an element of (the carrier of $K)^i$. Then $R + i \mapsto 0_K = R$ and $R = i \mapsto 0_K + R$.

Let K be a non empty loop structure and let p be a finite sequence of elements of the carrier of K. The functor $-p$ yields a finite sequence of elements of the carrier of K and is defined by:

(Def. 4) $-p = \mathrm{comp}K \cdot p$.

In the sequel K is a non empty loop structure, a is an element of K, p is a finite sequence of elements of the carrier of K, and R is an element of (the carrier of $K)^i$.

One can prove the following proposition

(30)[5] If $i \in \mathrm{dom}(-p)$ and $a = p(i)$, then $(-p)(i) = -a$.

Let us consider i, let K be a non empty loop structure, and let R be an element of (the carrier of $K)^i$. Then $-R$ is an element of (the carrier of $K)^i$.

The following propositions are true:

(31) If $j \in \mathrm{Seg}\, i$ and $a = R(j)$, then $(-R)(j) = -a$.

(32) $-\varepsilon_{(\text{the carrier of } K)} = \varepsilon_{(\text{the carrier of } K)}$.

(33) $-\langle a\rangle = \langle -a\rangle$.

(34) $-i \mapsto a = i \mapsto (-a)$.

(35) Let K be an Abelian right zeroed add-associative right complementable non empty loop structure and R be an element of (the carrier of $K)^i$. Then $R + -R = i \mapsto 0_K$ and $-R + R = i \mapsto 0_K$.

In the sequel K is a left zeroed right zeroed add-associative right complementable non empty loop structure and R, R_1, R_2 are elements of (the carrier of $K)^i$.

Next we state several propositions:

(36) If $R_1 + R_2 = i \mapsto 0_K$, then $R_1 = -R_2$ and $R_2 = -R_1$.

(37) $--R = R$.

(38) If $-R_1 = -R_2$, then $R_1 = R_2$.

(39) Let K be an Abelian right zeroed add-associative right complementable non empty loop structure and R, R_1, R_2 be elements of (the carrier of $K)^i$. If $R_1 + R = R_2 + R$ or $R_1 + R = R + R_2$, then $R_1 = R_2$.

(40) Let K be an Abelian right zeroed add-associative right complementable non empty loop structure and R_1, R_2 be elements of (the carrier of $K)^i$. Then $-(R_1 + R_2) = -R_1 + -R_2$.

Let K be a non empty loop structure and let p_1, p_2 be finite sequences of elements of the carrier of K. The functor $p_1 - p_2$ yielding a finite sequence of elements of the carrier of K is defined by:

[5] The proposition (29) has been removed.

(Def. 5) $p_1 - p_2 = (-_K)^\circ(p_1, p_2)$.

For simplicity, we use the following convention: K denotes a non empty loop structure, a_1, a_2 denote elements of K, p_1, p_2 denote finite sequences of elements of the carrier of K, and R_1, R_2 denote elements of (the carrier of $K)^i$.

The following proposition is true

(42)[6] If $i \in \mathrm{dom}(p_1 - p_2)$ and $a_1 = p_1(i)$ and $a_2 = p_2(i)$, then $(p_1 - p_2)(i) = a_1 - a_2$.

Let us consider i, let K be a non empty loop structure, and let R_1, R_2 be elements of (the carrier of $K)^i$. Then $R_1 - R_2$ is an element of (the carrier of $K)^i$.

One can prove the following propositions:

(43) If $j \in \mathrm{Seg}\, i$ and $a_1 = R_1(j)$ and $a_2 = R_2(j)$, then $(R_1 - R_2)(j) = a_1 - a_2$.

(44) $\varepsilon_{(\text{the carrier of } K)} - p_1 = \varepsilon_{(\text{the carrier of } K)}$ and $p_1 - \varepsilon_{(\text{the carrier of } K)} = \varepsilon_{(\text{the carrier of } K)}$.

(45) $\langle a_1 \rangle - \langle a_2 \rangle = \langle a_1 - a_2 \rangle$.

(46) $i \mapsto a_1 - i \mapsto a_2 = i \mapsto (a_1 - a_2)$.

(47) $R_1 - R_2 = R_1 + -R_2$.

(48) Let K be an add-associative right complementable left zeroed right zeroed non empty loop structure and R be an element of (the carrier of $K)^i$. Then $R - i \mapsto 0_K = R$.

(49) Let K be an Abelian left zeroed right zeroed non empty loop structure and R be an element of (the carrier of $K)^i$. Then $i \mapsto 0_K - R = -R$.

(50) Let K be a left zeroed right zeroed add-associative right complementable non empty loop structure and R_1, R_2 be elements of (the carrier of $K)^i$. Then $R_1 - -R_2 = R_1 + R_2$.

We adopt the following convention: K is an Abelian right zeroed add-associative right complementable non empty loop structure and R, R_1, R_2, R_3 are elements of (the carrier of $K)^i$.

The following propositions are true:

(51) $-(R_1 - R_2) = R_2 - R_1$.

(52) $-(R_1 - R_2) = -R_1 + R_2$.

(53) $R - R = i \mapsto 0_K$.

(54) If $R_1 - R_2 = i \mapsto 0_K$, then $R_1 = R_2$.

(55) $R_1 - R_2 - R_3 = R_1 - (R_2 + R_3)$.

(56) $R_1 + (R_2 - R_3) = (R_1 + R_2) - R_3$.

(57) $R_1 - (R_2 - R_3) = (R_1 - R_2) + R_3$.

(58) $R_1 = (R_1 + R) - R$.

(59) $R_1 = (R_1 - R) + R$.

For simplicity, we follow the rules: K denotes a non empty groupoid, a, a', a_1, a_2 denote elements of K, p denotes a finite sequence of elements of the carrier of K, and R denotes an element of (the carrier of $K)^i$.

The following two propositions are true:

(60) For all elements a, b of K holds $((\text{the multiplication of } K)^\circ(a, \mathrm{id}_{\text{the carrier of } K}))(b) = a \cdot b$.

(61) For all elements a, b of K holds $\cdot^a(b) = a \cdot b$.

[6] The proposition (41) has been removed.

Let K be a non empty groupoid, let p be a finite sequence of elements of the carrier of K, and let a be an element of K. The functor $a \cdot p$ yielding a finite sequence of elements of the carrier of K is defined as follows:

(Def. 6) $a \cdot p = {\cdot}^{a} \cdot p$.

One can prove the following proposition

(62) If $i \in \mathrm{dom}(a \cdot p)$ and $a' = p(i)$, then $(a \cdot p)(i) = a \cdot a'$.

Let us consider i, let K be a non empty groupoid, let R be an element of (the carrier of $K)^i$, and let a be an element of K. Then $a \cdot R$ is an element of (the carrier of $K)^i$.

Next we state several propositions:

(63) If $j \in \mathrm{Seg}\, i$ and $a' = R(j)$, then $(a \cdot R)(j) = a \cdot a'$.

(64) $a \cdot \varepsilon_{(\text{the carrier of } K)} = \varepsilon_{(\text{the carrier of } K)}$.

(65) $a \cdot \langle a_1 \rangle = \langle a \cdot a_1 \rangle$.

(66) $a_1 \cdot (i \mapsto a_2) = i \mapsto (a_1 \cdot a_2)$.

(67) Let K be an associative non empty groupoid, a_1, a_2 be elements of K, and R be an element of (the carrier of $K)^i$. Then $(a_1 \cdot a_2) \cdot R = a_1 \cdot (a_2 \cdot R)$.

We use the following convention: K is a distributive non empty double loop structure, a, a_1, a_2 are elements of K, and R, R_1, R_2 are elements of (the carrier of $K)^i$.

We now state several propositions:

(68) $(a_1 + a_2) \cdot R = a_1 \cdot R + a_2 \cdot R$.

(69) $a \cdot (R_1 + R_2) = a \cdot R_1 + a \cdot R_2$.

(70) Let K be a distributive commutative left unital non empty double loop structure and R be an element of (the carrier of $K)^i$. Then $\mathbf{1}_K \cdot R = R$.

(71) Let K be an add-associative right zeroed right complementable distributive non empty double loop structure and R be an element of (the carrier of $K)^i$. Then $0_K \cdot R = i \mapsto 0_K$.

(72) Let K be an add-associative right zeroed right complementable commutative left unital distributive non empty double loop structure and R be an element of (the carrier of $K)^i$. Then $(-\mathbf{1}_K) \cdot R = -R$.

Let M be a non empty groupoid and let p_1, p_2 be finite sequences of elements of the carrier of M. The functor $p_1 \bullet p_2$ yields a finite sequence of elements of the carrier of M and is defined by:

(Def. 7) $p_1 \bullet p_2 = (\text{the multiplication of } M)^{\circ}(p_1, p_2)$.

For simplicity, we adopt the following rules: K denotes a non empty groupoid, a_1, a_2 denote elements of K, p, p_1, p_2 denote finite sequences of elements of the carrier of K, and R_1, R_2 denote elements of (the carrier of $K)^i$.

One can prove the following proposition

(73) If $i \in \mathrm{dom}(p_1 \bullet p_2)$ and $a_1 = p_1(i)$ and $a_2 = p_2(i)$, then $(p_1 \bullet p_2)(i) = a_1 \cdot a_2$.

Let us consider i, let K be a non empty groupoid, and let R_1, R_2 be elements of (the carrier of $K)^i$. Then $R_1 \bullet R_2$ is an element of (the carrier of $K)^i$.

The following three propositions are true:

(74) If $j \in \mathrm{Seg}\, i$ and $a_1 = R_1(j)$ and $a_2 = R_2(j)$, then $(R_1 \bullet R_2)(j) = a_1 \cdot a_2$.

(75) $\varepsilon_{(\text{the carrier of } K)} \bullet p = \varepsilon_{(\text{the carrier of } K)}$ and $p \bullet \varepsilon_{(\text{the carrier of } K)} = \varepsilon_{(\text{the carrier of } K)}$.

(76) $\langle a_1 \rangle \bullet \langle a_2 \rangle = \langle a_1 \cdot a_2 \rangle$.

We follow the rules: K is a commutative non empty groupoid, p, q are finite sequences of elements of the carrier of K, and R_1, R_2 are elements of (the carrier of $K)^i$.

Next we state three propositions:

(77) $R_1 \bullet R_2 = R_2 \bullet R_1$.

(78) $p \bullet q = q \bullet p$.

(79) For every associative non empty groupoid K and for all elements R_1, R_2, R_3 of (the carrier of $K)^i$ holds $R_1 \bullet (R_2 \bullet R_3) = (R_1 \bullet R_2) \bullet R_3$.

We adopt the following convention: K is a commutative associative non empty groupoid, a, a_1, a_2 are elements of K, and R is an element of (the carrier of $K)^i$.

We now state three propositions:

(80) $i \mapsto a \bullet R = a \cdot R$ and $R \bullet i \mapsto a = a \cdot R$.

(81) $i \mapsto a_1 \bullet i \mapsto a_2 = i \mapsto (a_1 \cdot a_2)$.

(82) Let K be an associative non empty groupoid, a be an element of K, and R_1, R_2 be elements of (the carrier of $K)^i$. Then $a \cdot (R_1 \bullet R_2) = a \cdot R_1 \bullet R_2$.

We follow the rules: K is a commutative associative non empty groupoid, a is an element of K, and R, R_1, R_2 are elements of (the carrier of $K)^i$.

One can prove the following propositions:

(83) $a \cdot (R_1 \bullet R_2) = a \cdot R_1 \bullet R_2$ and $a \cdot (R_1 \bullet R_2) = R_1 \bullet a \cdot R_2$.

(84) $a \cdot R = i \mapsto a \bullet R$.

3. THE SUM OF FINITE NUMBER OF ELEMENTS

Let us observe that every non empty loop structure which is Abelian and right zeroed is also left zeroed.

Let K be an Abelian add-associative right zeroed right complementable non empty loop structure and let p be a finite sequence of elements of the carrier of K. Then $\sum p$ can be characterized by the condition:

(Def. 8) $\sum p =$ the addition of $K \circledast p$.

In the sequel K denotes an add-associative right zeroed right complementable non empty loop structure, a denotes an element of K, and p denotes a finite sequence of elements of the carrier of K.

One can prove the following propositions:

(87)[7] $\sum(p \frown \langle a \rangle) = \sum p + a$.

(89)[8] $\sum(\langle a \rangle \frown p) = a + \sum p$.

(92)[9] Let K be an Abelian add-associative right zeroed right complementable distributive non empty double loop structure, a be an element of K, and p be a finite sequence of elements of the carrier of K. Then $\sum(a \cdot p) = a \cdot \sum p$.

(93) For every non empty loop structure K and for every element R of (the carrier of $K)^0$ holds $\sum R = 0_K$.

[7] The propositions (85) and (86) have been removed.
[8] The proposition (88) has been removed.
[9] The propositions (90) and (91) have been removed.

In the sequel K denotes an Abelian add-associative right zeroed right complementable non empty loop structure, p denotes a finite sequence of elements of the carrier of K, and R_1, R_2 denote elements of (the carrier of $K)^i$.

Next we state three propositions:

(94) $\sum(-p) = -\sum p$.

(95) $\sum(R_1 + R_2) = \sum R_1 + \sum R_2$.

(96) $\sum(R_1 - R_2) = \sum R_1 - \sum R_2$.

4. THE PRODUCT OF FINITE NUMBER OF ELEMENTS

Let K be a non empty groupoid and let p be a finite sequence of elements of the carrier of K. The functor $\prod p$ yields an element of K and is defined by:

(Def. 9) $\prod p =$ the multiplication of $K \circledast p$.

The following propositions are true:

(98)[10] For every commutative left unital non empty double loop structure K holds $\prod(\varepsilon_{(\text{the carrier of } K)}) = \mathbf{1}_K$.

(99) For every non empty groupoid K and for every element a of K holds $\prod\langle a\rangle = a$.

(100) Let K be a commutative left unital non empty double loop structure, a be an element of K, and p be a finite sequence of elements of the carrier of K. Then $\prod(p \frown \langle a\rangle) = \prod p \cdot a$.

For simplicity, we use the following convention: K denotes a commutative associative left unital non empty double loop structure, a, a_1, a_2, a_3 denote elements of K, p_1, p_2 denote finite sequences of elements of the carrier of K, and R_1, R_2 denote elements of (the carrier of $K)^i$.

One can prove the following propositions:

(101) $\prod(p_1 \frown p_2) = \prod p_1 \cdot \prod p_2$.

(102) $\prod(\langle a\rangle \frown p_1) = a \cdot \prod p_1$.

(103) $\prod\langle a_1, a_2\rangle = a_1 \cdot a_2$.

(104) $\prod\langle a_1, a_2, a_3\rangle = a_1 \cdot a_2 \cdot a_3$.

(105) For every element R of (the carrier of $K)^0$ holds $\prod R = \mathbf{1}_K$.

(106) $\prod(i \mapsto \mathbf{1}_K) = \mathbf{1}_K$.

(107) Let K be an add-associative right zeroed right complementable Abelian commutative associative left unital distributive field-like non degenerated non empty double loop structure and p be a finite sequence of elements of the carrier of K. Then there exists k such that $k \in \operatorname{dom} p$ and $p(k) = 0_K$ if and only if $\prod p = 0_K$.

(108) $\prod((i + j) \mapsto a) = \prod(i \mapsto a) \cdot \prod(j \mapsto a)$.

(109) $\prod((i \cdot j) \mapsto a) = \prod(j \mapsto \prod(i \mapsto a))$.

(110) $\prod(i \mapsto (a_1 \cdot a_2)) = \prod(i \mapsto a_1) \cdot \prod(i \mapsto a_2)$.

(111) $\prod(R_1 \bullet R_2) = \prod R_1 \cdot \prod R_2$.

(112) $\prod(a \cdot R_1) = \prod(i \mapsto a) \cdot \prod R_1$.

[10] The proposition (97) has been removed.

5. The product of vectors

Let K be a non empty double loop structure and let p, q be finite sequences of elements of the carrier of K. The functor $p \cdot q$ yielding an element of K is defined as follows:

(Def. 10) $p \cdot q = \sum(p \bullet q)$.

The following propositions are true:

(113) Let K be a commutative associative left unital Abelian add-associative right zeroed right complementable non empty double loop structure and a, b be elements of K. Then $\langle a \rangle \cdot \langle b \rangle = a \cdot b$.

(114) Let K be a commutative associative left unital Abelian add-associative right zeroed right complementable non empty double loop structure and a_1, a_2, b_1, b_2 be elements of K. Then $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 \cdot b_1 + a_2 \cdot b_2$.

(115) For all finite sequences p, q of elements of the carrier of K holds $p \cdot q = q \cdot p$.

Acknowledgments

I would like to thank Czesław Byliński for his help.

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Received December 29, 1992

Published January 2, 2004