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Properties of Fuzzy Relation

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Summary. In this article, we introduce four fuzzy relations and the composition, and some useful properties are shown by them. In section 2, the definition of converse relation R^{-1} of fuzzy relation R and properties concerning it are described. In the next section, we define the composition of the fuzzy relation and show some properties. In the final section we describe the identity relation, the universe relation and the zero relation.

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The articles [9], [2], [12], [1], [10], [11], [13], [5], [4], [3], [6], [7], and [8] provide the notation and terminology for this paper.

1. Basic Properties of the Membership Function

We follow the rules: x, y, z denote sets and C_1, C_2, C_3 denote non empty sets.

Let C_1 be a non empty set and let F be a membership function of C_1. Observe that $\operatorname{rng} F$ is non empty.

One can prove the following four propositions:

(1) Let F be a membership function of C_1. Then $\operatorname{rng} F$ is bounded and for every x such that $x \in \operatorname{dom} F$ holds $F(x) \leq \sup \operatorname{rng} F$ and for every x such that $x \in \operatorname{dom} F$ holds $F(x) \geq \inf \operatorname{rng} F$.

(2) For all membership functions F, G of C_1 such that for every x such that $x \in C_1$ holds $F(x) \leq G(x)$ holds $\sup \operatorname{rng} F \leq \sup \operatorname{rng} G$.

(3) For every membership function f of C_1, C_2 and for every element c of $[: C_1, C_2 :]$ holds $0 \leq f(c)$ and $f(c) \leq 1$.

(4) For every membership function f of C_1, C_2 and for all x, y such that $\langle x, y \rangle \in [: C_1, C_2 :]$ holds $0 \leq f(\langle x, y \rangle)$ and $f(\langle x, y \rangle) \leq 1$.

2. Definition of Converse Fuzzy Relation and some Properties

Let C_1, C_2 be non empty sets and let h be a membership function of C_2, C_1. The functor converse h yielding a membership function of C_1, C_2 is defined by:

(Def. 1) For all x, y such that $\langle x, y \rangle \in [: C_1, C_2 :]$ holds $(\operatorname{converse} h)(\langle x, y \rangle) = h(\langle y, x \rangle)$.

Let C_1, C_2 be non empty sets, let f be a membership function of C_2, C_1, and let R be a fuzzy relation of C_2, C_1, f. The functor R^{-1} yields a fuzzy relation of C_1, C_2, converse f and is defined as follows:

(Def. 2) $R^{-1} = [: [: C_1, C_2 :], (\text{converse} f)^{\circ}[: C_1, C_2 :] :]$.

Next we state a number of propositions:

(5) For every membership function f of C_1, C_2 holds converse converse $f = f$.

(6) For every membership function f of C_1, C_2 and for every fuzzy relation R of C_1, C_2, f holds $(R^{-1})^{-1} = R$.

(7) For every membership function f of C_1, C_2 holds 1-minus converse $f =$ converse 1-minus f.

(8) For every membership function f of C_1, C_2 and for every fuzzy relation R of C_1, C_2, f holds $(R^{-1})^{\mathrm{c}} = (R^{\mathrm{c}})^{-1}$.

(9) For all membership functions f, g of C_1, C_2 holds converse $\max(f, g) = \max(\text{converse} f, \text{converse} g)$.

(10) Let f, g be membership functions of C_1, C_2, R be a fuzzy relation of C_1, C_2, f, and S be a fuzzy relation of C_1, C_2, g. Then $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

(11) For all membership functions f, g of C_1, C_2 holds converse $\min(f, g) = \min(\text{converse} f, \text{converse} g)$.

(12) Let f, g be membership functions of C_1, C_2, R be a fuzzy relation of C_1, C_2, f, and S be a fuzzy relation of C_1, C_2, g. Then $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$.

(13) Let f, g be membership functions of C_1, C_2 and given x, y. If $x \in C_1$ and $y \in C_2$, then if $f(\langle x, y \rangle) \leq g(\langle x, y \rangle)$, then $(\text{converse} f)(\langle y, x \rangle) \leq (\text{converse} g)(\langle y, x \rangle)$.

(14) Let f, g be membership functions of C_1, C_2, R be a fuzzy relation of C_1, C_2, f, and S be a fuzzy relation of C_1, C_2, g. If $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$.

(15) For all membership functions f, g of C_1, C_2 holds converse $\min(f, \text{1-minus} g) = \min(\text{converse} f, \text{1-minus converse} g)$.

(16) Let f, g be membership functions of C_1, C_2, R be a fuzzy relation of C_1, C_2, f, and S be a fuzzy relation of C_1, C_2, g. Then $(R \setminus S)^{-1} = R^{-1} \setminus S^{-1}$.

(17) For all membership functions f, g of C_1, C_2 holds converse $\max(\min(f, \text{1-minus} g), \min(\text{1-minus} f, g)) = \max(\min(\text{converse} f, \text{1-minus converse} g), \min(\text{1-minus converse} f, \text{converse} g))$.

(18) Let f, g be membership functions of C_1, C_2, R be a fuzzy relation of C_1, C_2, f, and S be a fuzzy relation of C_1, C_2, g. Then $(R \dot{-} S)^{-1} = R^{-1} \dot{-} S^{-1}$.

3. Definition of the Composition and some Properties

Let C_1, C_2, C_3 be non empty sets, let h be a membership function of C_1, C_2, let g be a membership function of C_2, C_3, and let x, z be sets. Let us assume that $x \in C_1$ and $z \in C_3$. The functor $\min(h, g, x, z)$ yields a membership function of C_2 and is defined by:

(Def. 3) For every element y of C_2 holds $(\min(h, g, x, z))(y) = \min(h(\langle x, y \rangle), g(\langle y, z \rangle))$.

Let C_1, C_2, C_3 be non empty sets, let h be a membership function of C_1, C_2, and let g be a membership function of C_2, C_3. The functor $h\,g$ yielding a membership function of C_1, C_3 is defined by:

(Def. 4) For all x, z such that $\langle x, z \rangle \in [: C_1, C_3 :]$ holds $(h\,g)(\langle x, z \rangle) = \sup \operatorname{rng} \min(h, g, x, z)$.

Let C_1, C_2, C_3 be non empty sets, let f be a membership function of C_1, C_2, let g be a membership function of C_2, C_3, let R be a fuzzy relation of C_1, C_2, f, and let S be a fuzzy relation of C_2, C_3, g. The functor $R\,S$ yielding a fuzzy relation of C_1, C_3, $f\,g$ is defined by:

(Def. 5) $R\,S = [:[:C_1, C_3:], (f\,g)^{\circ}[:C_1, C_3:]:]$.

The following propositions are true:

(19) For every membership function f of C_1, C_2 and for all membership functions g, h of C_2, C_3 holds $f\ \max(g,h) = \max(f\,g, f\,h)$.

(20) Let f be a membership function of C_1, C_2, g, h be membership functions of C_2, C_3, R be a fuzzy relation of C_1, C_2, f, S be a fuzzy relation of C_2, C_3, g, and T be a fuzzy relation of C_2, C_3, h. Then $R\,(S \cup T) = R\,S \cup R\,T$.

(21) For all membership functions f, g of C_1, C_2 and for every membership function h of C_2, C_3 holds $\max(f,g)\,h = \max(f\,h, g\,h)$.

(22) Let f, g be membership functions of C_1, C_2, h be a membership function of C_2, C_3, R be a fuzzy relation of C_1, C_2, f, S be a fuzzy relation of C_1, C_2, g, and T be a fuzzy relation of C_2, C_3, h. Then $(R \cup S)\,T = R\,T \cup S\,T$.

(23) Let f be a membership function of C_1, C_2, g, h be membership functions of C_2, C_3, and x, z be sets. If $x \in C_1$ and $z \in C_3$, then $(f\ \min(g,h))(\langle x, z\rangle) \leq (\min(f\,g, f\,h))(\langle x, z\rangle)$.

(24) Let f be a membership function of C_1, C_2, g, h be membership functions of C_2, C_3, R be a fuzzy relation of C_1, C_2, f, S be a fuzzy relation of C_2, C_3, g, and T be a fuzzy relation of C_2, C_3, h. Then $R\,(S \cap T) \subseteq (R\,S) \cap (R\,T)$.

(25) Let f, g be membership functions of C_1, C_2, h be a membership function of C_2, C_3, and x, z be sets. If $x \in C_1$ and $z \in C_3$, then $(\min(f,g)\,h)(\langle x, z\rangle) \leq (\min(f\,h, g\,h))(\langle x, z\rangle)$.

(26) Let f, g be membership functions of C_1, C_2, h be a membership function of C_2, C_3, R be a fuzzy relation of C_1, C_2, f, S be a fuzzy relation of C_1, C_2, g, and T be a fuzzy relation of C_2, C_3, h. Then $(R \cap S)\,T \subseteq (R\,T) \cap (S\,T)$.

(27) For every membership function f of C_1, C_2 and for every membership function g of C_2, C_3 holds $\mathrm{converse} f\ g = \mathrm{converse} g\ \mathrm{converse} f$.

(28) Let f be a membership function of C_1, C_2, g be a membership function of C_2, C_3, R be a fuzzy relation of C_1, C_2, f, and S be a fuzzy relation of C_2, C_3, g. Then $(R\,S)^{-1} = S^{-1}\,R^{-1}$.

(29) Let f, g be membership functions of C_1, C_2, h, k be membership functions of C_2, C_3, and x, z be sets. Suppose $x \in C_1$ and $z \in C_3$ and for every set y such that $y \in C_2$ holds $f(\langle x, y\rangle) \leq g(\langle x, y\rangle)$ and $h(\langle y, z\rangle) \leq k(\langle y, z\rangle)$. Then $(f\,h)(\langle x, z\rangle) \leq (g\,k)(\langle x, z\rangle)$.

(30) Let f, g be membership functions of C_1, C_2, h, k be membership functions of C_2, C_3, R be a fuzzy relation of C_1, C_2, f, S be a fuzzy relation of C_1, C_2, g, T be a fuzzy relation of C_2, C_3, h, and W be a fuzzy relation of C_2, C_3, k. If $R \subseteq S$ and $T \subseteq W$, then $R\,T \subseteq S\,W$.

4. Definition of Identity Relation and Properties of Universe and Zero Relation

Let C_1, C_2 be non empty sets. The functor $\mathrm{Imf}(C_1,C_2)$ yielding a membership function of C_1, C_2 is defined as follows:

(Def. 6) For all x, y such that $\langle x, y\rangle \in [:C_1, C_2:]$ holds if $x = y$, then $(\mathrm{Imf}(C_1,C_2))(\langle x, y\rangle) = 1$ and if $x \neq y$, then $(\mathrm{Imf}(C_1,C_2))(\langle x, y\rangle) = 0$.

Next we state several propositions:

(31) For every element c of $[:C_1, C_2:]$ holds $(\mathrm{Zmf}(C_1,C_2))(c) = 0$ and $(\mathrm{Umf}(C_1,C_2))(c) = 1$.

(32) For all x, y such that $\langle x, y\rangle \in [:C_1, C_2:]$ holds $(\mathrm{Zmf}(C_1,C_2))(\langle x, y\rangle) = 0$ and $(\mathrm{Umf}(C_1,C_2))(\langle x, y\rangle) = 1$.

(33) Let f be a membership function of C_2, C_3, O_1 be a zero relation of C_1, C_2, O_2 be a zero relation of C_1, C_3, and R be a fuzzy relation of C_2, C_3, f. Then $O_1\,R = O_2$.

(34) For every membership function f of C_1, C_2 holds $f\ \mathrm{Zmf}(C_2,C_3) = \mathrm{Zmf}(C_1,C_3)$.

(35) Let f be a membership function of C_1, C_2, O_1 be a zero relation of C_2, C_3, O_2 be a zero relation of C_1, C_3, and R be a fuzzy relation of C_1, C_2, f. Then $R\,O_1 = O_2$.

(36) For every membership function f of C_1, C_1 holds $f\ \mathrm{Zmf}(C_1,C_1) = \mathrm{Zmf}(C_1,C_1)\,f$.

(37) Let f be a membership function of C_1, C_1, O be a zero relation of C_1, C_1, and R be a fuzzy relation of C_1, C_1, f. Then $R\,O = O\,R$.

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