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Basic Properties of Functor Structures

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Summary. This article presents some theorems about functor structures. We start with some basic lemmata concerning the composition of functor structures. Then, two theorems about the restriction operator are formulated. Later we show two theorems concerning the properties 'full' and 'faithful' of functor structures which are equivalent to the 'onto' and 'one-to-one' properties of their morphmaps, respectively. Furthermore, we prove some theorems about the inversion of functor structures.

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The articles [9], [6], [15], [16], [3], [5], [4], [2], [10], [11], [8], [7], [12], [13], [1], and [14] provide the notation and terminology for this paper.

1. Definitions

In this paper X, Y are sets and Z is a non empty set.

Let us note that there exists a non empty category structure which is transitive and reflexive and has units.

Let A be a non empty reflexive category structure. Observe that there exists a substructure of A which is non empty and reflexive.

Let C_1, C_2 be non empty reflexive category structures, let F be a feasible functor structure from C_1 to C_2, and let A be a non empty reflexive substructure of C_1. One can verify that $F{\restriction}A$ is feasible.

2. Theorems about sets and functions

Next we state four propositions:

(1) For every set X holds id_X is onto.

(2) Let A be a non empty set, B, C be non empty subsets of A, and D be a non empty subset of B. If $C = D$, then $\overset{C}{\hookrightarrow} = (\overset{B}{\hookrightarrow}) \cdot (\overset{D}{\hookrightarrow})$.

(3) For every function f from X into Y such that f is bijective holds f^{-1} is a function from Y into X.

(4) Let f be a function from X into Y and g be a function from Y into Z. Suppose f is bijective and g is bijective. Then there exists a function h from X into Z such that $h = g \cdot f$ and h is bijective.

3. Theorems about the Composition of Functor Structures

The following propositions are true:

(5) Let A be a non empty reflexive category structure, B be a non empty reflexive substructure of A, C be a non empty substructure of A, and D be a non empty substructure of B. If $C = D$, then $\overset{C}{\hookrightarrow} = (\overset{B}{\hookrightarrow}) \cdot (\overset{D}{\hookrightarrow})$.

(6) Let A, B be non empty category structures and F be a functor structure from A to B. Suppose F is bijective. Then the object map of F is bijective and the morphism map of F is "1-1".

(7) Let C_1 be a non empty graph, C_2, C_3 be non empty reflexive graphs, F be a feasible functor structure from C_1 to C_2, and G be a functor structure from C_2 to C_3. If F is one-to-one and G is one-to-one, then $G \cdot F$ is one-to-one.

(8) Let C_1 be a non empty graph, C_2, C_3 be non empty reflexive graphs, F be a feasible functor structure from C_1 to C_2, and G be a functor structure from C_2 to C_3. If F is faithful and G is faithful, then $G \cdot F$ is faithful.

(9) Let C_1 be a non empty graph, C_2, C_3 be non empty reflexive graphs, F be a feasible functor structure from C_1 to C_2, and G be a functor structure from C_2 to C_3. If F is onto and G is onto, then $G \cdot F$ is onto.

(10) Let C_1 be a non empty graph, C_2, C_3 be non empty reflexive graphs, F be a feasible functor structure from C_1 to C_2, and G be a functor structure from C_2 to C_3. If F is full and G is full, then $G \cdot F$ is full.

(11) Let C_1 be a non empty graph, C_2, C_3 be non empty reflexive graphs, F be a feasible functor structure from C_1 to C_2, and G be a functor structure from C_2 to C_3. If F is injective and G is injective, then $G \cdot F$ is injective.

(12) Let C_1 be a non empty graph, C_2, C_3 be non empty reflexive graphs, F be a feasible functor structure from C_1 to C_2, and G be a functor structure from C_2 to C_3. If F is surjective and G is surjective, then $G \cdot F$ is surjective.

(13) Let C_1 be a non empty graph, C_2, C_3 be non empty reflexive graphs, F be a feasible functor structure from C_1 to C_2, and G be a functor structure from C_2 to C_3. If F is bijective and G is bijective, then $G \cdot F$ is bijective.

4. Theorems about the Restriction and Inclusion Operator

The following three propositions are true:

(14) Let A, I be non empty reflexive category structures, B be a non empty reflexive substructure of A, C be a non empty substructure of A, and D be a non empty substructure of B. Suppose $C = D$. Let F be a functor structure from A to I. Then $F \restriction C = F \restriction B \restriction D$.

(15) Let C_1, C_2, C_3 be non empty reflexive category structures, F be a feasible functor structure from C_1 to C_2, G be a functor structure from C_2 to C_3, and A be a non empty reflexive substructure of C_1. Then $(G \cdot F) \restriction A = G \cdot (F \restriction A)$.

(17)[1] Let A be a non empty category structure and B be a non empty substructure of A. Then B is full if and only if $\overset{B}{\hookrightarrow}$ is full.

[1] The proposition (16) has been removed.

5. Theorems about 'full' and 'faithful' functor structures

We now state two propositions:

(18) Let C_1, C_2 be non empty category structures and F be a precovariant functor structure from C_1 to C_2. Then F is full if and only if for all objects o_1, o_2 of C_1 holds Morph-Map$_F(o_1,o_2)$ is onto.

(19) Let C_1, C_2 be non empty category structures and F be a precovariant functor structure from C_1 to C_2. Then F is faithful if and only if for all objects o_1, o_2 of C_1 holds Morph-Map$_F(o_1,o_2)$ is one-to-one.

6. Theorems about the inversion of functor structures

We now state several propositions:

(20) For every transitive non empty category structure A with units holds $(\mathrm{id}_A)^{-1} = \mathrm{id}_A$.

(21) Let A, B be transitive reflexive non empty category structures with units and F be a feasible functor structure from A to B. Suppose F is bijective. Let G be a feasible functor structure from B to A. If the functor structure of $G = F^{-1}$, then $F \cdot G = \mathrm{id}_B$.

(22) Let A, B be transitive reflexive non empty category structures with units and F be a feasible functor structure from A to B. If F is bijective, then $F^{-1} \cdot F = \mathrm{id}_A$.

(23) Let A, B be transitive reflexive non empty category structures with units and F be a feasible functor structure from A to B. If F is bijective, then $(F^{-1})^{-1} =$ the functor structure of F.

(24) Let A, B, C be transitive reflexive non empty category structures with units, G be a feasible functor structure from A to B, F be a feasible functor structure from B to C, G_1 be a feasible functor structure from B to A, and F_1 be a feasible functor structure from C to B. Suppose that

(i) F is bijective,

(ii) G is bijective,

(iii) F_1 is bijective,

(iv) G_1 is bijective,

(v) the functor structure of $G_1 = G^{-1}$, and

(vi) the functor structure of $F_1 = F^{-1}$.

Then $(F \cdot G)^{-1} = G_1 \cdot F_1$.

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