JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

The Modification of a Function by a Function and the Iteration of the Composition of a Function

Czesław Byliński
Warsaw University
Białystok

Summary. In the article we introduce some operations on functions. We define the natural ordering relation on functions. The fact that a function f is less than a function g we denote by $f \leq g$ and we define by graph$f \subseteq$ graph$f$. In the sequel we define the modifications of a function f by a function g denoted $f {+\cdot} g$ and the n-th iteration of the composition of a function f denoted by f^n. We prove some propositions related to the introduced notions.

MML Identifier: FUNCT_4.

WWW: http://mizar.org/JFM/Vol2/funct_4.html

The articles [6], [4], [7], [8], [1], [9], [2], [3], and [5] provide the notation and terminology for this paper.

We adopt the following rules: a, b, x, x', x_1, x_2, y, y', y_1, y_2, z, X, X', Y, Y', Z, Z' denote sets, A, D, D' denote non empty sets, and f, g, h denote functions.

One can prove the following propositions:

(1) If for every z such that $z \in Z$ there exist x, y such that $z = \langle x, y \rangle$, then there exist X, Y such that $Z \subseteq [: X, Y :]$.

(2) $g \cdot f = (g \restriction \operatorname{rng} f) \cdot f$.

(3) $\emptyset = \emptyset \longmapsto a$.

(4) $\mathrm{id}_X \subseteq \mathrm{id}_Y$ iff $X \subseteq Y$.

(5) If $X \subseteq Y$, then $X \longmapsto a \subseteq Y \longmapsto a$.

(6) If $X \longmapsto a \subseteq Y \longmapsto b$, then $X \subseteq Y$.

(7) If $X \neq \emptyset$ and $X \longmapsto a \subseteq Y \longmapsto b$, then $a = b$.

(8) If $x \in \operatorname{dom} f$, then $\{x\} \longmapsto f(x) \subseteq f$.

Let us consider f, g. We introduce $f \leq g$ as a synonym of $f \subseteq g$.

One can prove the following two propositions:

(9) $Y \restriction f \restriction X \leq f$.

(10) If $f \leq g$, then $Y \restriction f \restriction X \leq Y \restriction g \restriction X$.

Let us consider f, g. The functor $f {+\cdot} g$ yields a function and is defined as follows:

(Def. 1) $\mathrm{dom}(f{+\cdot}g) = \mathrm{dom}\, f \cup \mathrm{dom}\, g$ and for every x such that $x \in \mathrm{dom}\, f \cup \mathrm{dom}\, g$ holds if $x \in \mathrm{dom}\, g$, then $(f{+\cdot}g)(x) = g(x)$ and if $x \notin \mathrm{dom}\, g$, then $(f{+\cdot}g)(x) = f(x)$.

Let us notice that the functor $f{+\cdot}g$ is idempotent.

The following propositions are true:

(11) $\mathrm{dom}\, f \subseteq \mathrm{dom}(f{+\cdot}g)$ and $\mathrm{dom}\, g \subseteq \mathrm{dom}(f{+\cdot}g)$.

(12) If $x \notin \mathrm{dom}\, g$, then $(f{+\cdot}g)(x) = f(x)$.

(13) $x \in \mathrm{dom}(f{+\cdot}g)$ iff $x \in \mathrm{dom}\, f$ or $x \in \mathrm{dom}\, g$.

(14) If $x \in \mathrm{dom}\, g$, then $(f{+\cdot}g)(x) = g(x)$.

(15) $(f{+\cdot}g){+\cdot}h = f{+\cdot}(g{+\cdot}h)$.

(16) If $f \approx g$ and $x \in \mathrm{dom}\, f$, then $(f{+\cdot}g)(x) = f(x)$.

(17) If $\mathrm{dom}\, f$ misses $\mathrm{dom}\, g$ and $x \in \mathrm{dom}\, f$, then $(f{+\cdot}g)(x) = f(x)$.

(18) $\mathrm{rng}(f{+\cdot}g) \subseteq \mathrm{rng}\, f \cup \mathrm{rng}\, g$.

(19) $\mathrm{rng}\, g \subseteq \mathrm{rng}(f{+\cdot}g)$.

(20) If $\mathrm{dom}\, f \subseteq \mathrm{dom}\, g$, then $f{+\cdot}g = g$.

(21) $\emptyset{+\cdot}f = f$.

(22) $f{+\cdot}\emptyset = f$.

(23) $\mathrm{id}_X{+\cdot}\mathrm{id}_Y = \mathrm{id}_{X \cup Y}$.

(24) $(f{+\cdot}g){\restriction}\mathrm{dom}\, g = g$.

(25) $(f{+\cdot}g){\restriction}(\mathrm{dom}\, f \setminus \mathrm{dom}\, g) \subseteq f$.

(26) $g \subseteq f{+\cdot}g$.

(27) If $f \approx g{+\cdot}h$, then $f{\restriction}(\mathrm{dom}\, f \setminus \mathrm{dom}\, h) \approx g$.

(28) If $f \approx g{+\cdot}h$, then $f \approx h$.

(29) $f \approx g$ iff $f \subseteq f{+\cdot}g$.

(30) $f{+\cdot}g \subseteq f \cup g$.

(31) $f \approx g$ iff $f \cup g = f{+\cdot}g$.

(32) If $\mathrm{dom}\, f$ misses $\mathrm{dom}\, g$, then $f \cup g = f{+\cdot}g$.

(33) If $\mathrm{dom}\, f$ misses $\mathrm{dom}\, g$, then $f \subseteq f{+\cdot}g$.

(34) If $\mathrm{dom}\, f$ misses $\mathrm{dom}\, g$, then $(f{+\cdot}g){\restriction}\mathrm{dom}\, f = f$.

(35) $f \approx g$ iff $f{+\cdot}g = g{+\cdot}f$.

(36) If $\mathrm{dom}\, f$ misses $\mathrm{dom}\, g$, then $f{+\cdot}g = g{+\cdot}f$.

(37) For all partial functions f, g from X to Y such that g is total holds $f{+\cdot}g = g$.

(38) For all functions f, g from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $f{+\cdot}g = g$.

(39) For all functions f, g from X into X holds $f{+\cdot}g = g$.

(40) For all functions f, g from X into D holds $f{+\cdot}g = g$.

(41) For all partial functions f, g from X to Y holds $f{+\cdot}g$ is a partial function from X to Y.

Let us consider f. The functor $\curvearrowleft f$ yielding a function is defined by the conditions (Def. 2).

(Def. 2)(i) For every x holds $x \in \mathrm{dom} \curvearrowleft f$ iff there exist y, z such that $x = \langle z, y \rangle$ and $\langle y, z \rangle \in \mathrm{dom} f$, and

(ii) for all y, z such that $\langle y, z \rangle \in \mathrm{dom} f$ holds $(\curvearrowleft f)(\langle z, y \rangle) = f(\langle y, z \rangle)$.

We now state a number of propositions:

(42) $\mathrm{rng} \curvearrowleft f \subseteq \mathrm{rng} f$.

(43) $\langle x, y \rangle \in \mathrm{dom} f$ iff $\langle y, x \rangle \in \mathrm{dom} \curvearrowleft f$.

(44) If $\langle y, x \rangle \in \mathrm{dom} \curvearrowleft f$, then $(\curvearrowleft f)(\langle y, x \rangle) = f(\langle x, y \rangle)$.

(45) There exist X, Y such that $\mathrm{dom} \curvearrowleft f \subseteq [\!: X, Y :\!]$.

(46) If $\mathrm{dom} f \subseteq [\!: X, Y :\!]$, then $\mathrm{dom} \curvearrowleft f \subseteq [\!: Y, X :\!]$.

(47) If $\mathrm{dom} f = [\!: X, Y :\!]$, then $\mathrm{dom} \curvearrowleft f = [\!: Y, X :\!]$.

(48) If $\mathrm{dom} f \subseteq [\!: X, Y :\!]$, then $\mathrm{rng} \curvearrowleft f = \mathrm{rng} f$.

(49) For every partial function f from $[\!: X, Y :\!]$ to Z holds $\curvearrowleft f$ is a partial function from $[\!: Y, X :\!]$ to Z.

(50) For every function f from $[\!: X, Y :\!]$ into Z such that $Z \neq \emptyset$ holds $\curvearrowleft f$ is a function from $[\!: Y, X :\!]$ into Z.

(51) For every function f from $[\!: X, Y :\!]$ into D holds $\curvearrowleft f$ is a function from $[\!: Y, X :\!]$ into D.

(52) $\curvearrowleft \curvearrowleft f \subseteq f$.

(53) If $\mathrm{dom} f \subseteq [\!: X, Y :\!]$, then $\curvearrowleft \curvearrowleft f = f$.

(54) For every partial function f from $[\!: X, Y :\!]$ to Z holds $\curvearrowleft \curvearrowleft f = f$.

(55) For every function f from $[\!: X, Y :\!]$ into Z such that $Z \neq \emptyset$ holds $\curvearrowleft \curvearrowleft f = f$.

(56) For every function f from $[\!: X, Y :\!]$ into D holds $\curvearrowleft \curvearrowleft f = f$.

Let us consider f, g. The functor $|\!: f, g :\!|$ yielding a function is defined by the conditions (Def. 3).

(Def. 3)(i) For every z holds $z \in \mathrm{dom} |\!: f, g :\!|$ iff there exist x, y, x', y' such that $z = \langle \langle x, x' \rangle, \langle y, y' \rangle \rangle$ and $\langle x, y \rangle \in \mathrm{dom} f$ and $\langle x', y' \rangle \in \mathrm{dom} g$, and

(ii) for all x, y, x', y' such that $\langle x, y \rangle \in \mathrm{dom} f$ and $\langle x', y' \rangle \in \mathrm{dom} g$ holds $|\!: f, g :\!|(\langle \langle x, x' \rangle, \langle y, y' \rangle \rangle) = \langle f(\langle x, y \rangle), g(\langle x', y' \rangle) \rangle$.

One can prove the following propositions:

(57) $\langle \langle x, x' \rangle, \langle y, y' \rangle \rangle \in \mathrm{dom} |\!: f, g :\!|$ iff $\langle x, y \rangle \in \mathrm{dom} f$ and $\langle x', y' \rangle \in \mathrm{dom} g$.

(58) If $\langle \langle x, x' \rangle, \langle y, y' \rangle \rangle \in \mathrm{dom} |\!: f, g :\!|$, then $|\!: f, g :\!|(\langle \langle x, x' \rangle, \langle y, y' \rangle \rangle) = \langle f(\langle x, y \rangle), g(\langle x', y' \rangle) \rangle$.

(59) $\mathrm{rng} |\!: f, g :\!| \subseteq [\!: \mathrm{rng} f, \mathrm{rng} g :\!]$.

(60) If $\mathrm{dom} f \subseteq [\!: X, Y :\!]$ and $\mathrm{dom} g \subseteq [\!: X', Y' :\!]$, then $\mathrm{dom} |\!: f, g :\!| \subseteq [\!: [\!: X, X' :\!], [\!: Y, Y' :\!] :\!]$.

(61) If $\mathrm{dom} f = [\!: X, Y :\!]$ and $\mathrm{dom} g = [\!: X', Y' :\!]$, then $\mathrm{dom} |\!: f, g :\!| = [\!: [\!: X, X' :\!], [\!: Y, Y' :\!] :\!]$.

(62) Let f be a partial function from $[\!: X, Y :\!]$ to Z and g be a partial function from $[\!: X', Y' :\!]$ to Z'. Then $|\!: f, g :\!|$ is a partial function from $[\!: [\!: X, X' :\!], [\!: Y, Y' :\!] :\!]$ to $[\!: Z, Z' :\!]$.

(63) Let f be a function from $[\!: X, Y :\!]$ into Z and g be a function from $[\!: X', Y' :\!]$ into Z'. If $Z \neq \emptyset$ and $Z' \neq \emptyset$, then $|\!: f, g :\!|$ is a function from $[\!: [\!: X, X' :\!], [\!: Y, Y' :\!] :\!]$ into $[\!: Z, Z' :\!]$.

(64) Let f be a function from $[:X, Y:]$ into D and g be a function from $[:X', Y':]$ into D'. Then $|:f, g:|$ is a function from $[:[:X, X':], [:Y, Y':]:]$ into $[:D, D':]$.

Let x, y, a, b be sets. The functor $[x \longmapsto a, y \longmapsto b]$ yields a set and is defined as follows:

(Def. 4) $[x \longmapsto a, y \longmapsto b] = (\{x\} \longmapsto a) {+\cdot} (\{y\} \longmapsto b)$.

Let x, y, a, b be sets. Observe that $[x \longmapsto a, y \longmapsto b]$ is function-like and relation-like.

One can prove the following four propositions:

(65) $\mathrm{dom}[x_1 \longmapsto y_1, x_2 \longmapsto y_2] = \{x_1, x_2\}$ and $\mathrm{rng}[x_1 \longmapsto y_1, x_2 \longmapsto y_2] \subseteq \{y_1, y_2\}$.

(66) If $x_1 \neq x_2$, then $[x_1 \longmapsto y_1, x_2 \longmapsto y_2](x_1) = y_1$ and $[x_1 \longmapsto y_1, x_2 \longmapsto y_2](x_2) = y_2$.

(67) If $x_1 \neq x_2$, then $\mathrm{rng}[x_1 \longmapsto y_1, x_2 \longmapsto y_2] = \{y_1, y_2\}$.

(68) $[x_1 \longmapsto y, x_2 \longmapsto y] = \{x_1, x_2\} \longmapsto y$.

Let us consider A, x_1, x_2 and let y_1, y_2 be elements of A. Then $[x_1 \longmapsto y_1, x_2 \longmapsto y_2]$ is a function from $\{x_1, x_2\}$ into A.

Next we state four propositions:

(69) For all sets a, b, c, d and for every function g such that $\mathrm{dom}\, g = \{a, b\}$ and $g(a) = c$ and $g(b) = d$ holds $g = [a \longmapsto c, b \longmapsto d]$.

(70) For all sets x, y holds $\{x\} \longmapsto y = \{\langle x, y \rangle\}$.

(71) For all sets a, b, c, d such that $a \neq c$ holds $[a \longmapsto b, c \longmapsto d] = \{\langle a, b \rangle, \langle c, d \rangle\}$.

(72) For all sets a, b, x, y, x', y' such that $a \neq b$ and $[a \longmapsto x, b \longmapsto y] = [a \longmapsto x', b \longmapsto y']$ holds $x = x'$ and $y = y'$.

REFERENCES

[1] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[2] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[3] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[4] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[5] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funcop_1.html.

[6] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[7] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[8] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[9] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

Received March 1, 1990

Published January 2, 2004