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Functions from a Set to a Set

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Summary. The article is a continuation of [1]. We define the following concepts: a function from a set X into a set Y, denoted by "Function of X,Y", the set of all functions from a set X into a set Y, denoted by Funcs(X,Y), and the permutation of a set (mode Permutation of X, where X is a set). Theorems and schemes included in the article are reformulations of the theorems of [1] in the new terminology. Also some basic facts about functions of two variables are proved.

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The articles [4], [3], [5], [6], [7], [1], and [2] provide the notation and terminology for this paper.

1. Functions from a Set to a Set

In this paper P, Q, X, Y, Z, x, x_1, x_2, y, z are sets.

Let us consider X, Y and let R be a relation between X and Y. We say that R is quasi total if and only if:

(Def. 1)(i) $X = \mathrm{dom}\, R$ if if $Y = \emptyset$, then $X = \emptyset$,

(ii) $R = \emptyset$, otherwise.

Let us consider X, Y. Observe that there exists a relation between X and Y which is quasi total and function-like.

Let us consider X, Y. One can verify that every partial function from X to Y which is total is also quasi total.

Let us consider X, Y. A function from X into Y is a quasi total function-like relation between X and Y.

We now state several propositions:

(3)[1] Every function f is a function from $\mathrm{dom}\, f$ into $\mathrm{rng}\, f$.

(4) For every function f such that $\mathrm{rng}\, f \subseteq Y$ holds f is a function from $\mathrm{dom}\, f$ into Y.

(5) For every function f such that $\mathrm{dom}\, f = X$ and for every x such that $x \in X$ holds $f(x) \in Y$ holds f is a function from X into Y.

(6) For every function f from X into Y such that $Y \neq \emptyset$ and $x \in X$ holds $f(x) \in \mathrm{rng}\, f$.

(7) For every function f from X into Y such that $Y \neq \emptyset$ and $x \in X$ holds $f(x) \in Y$.

[1] The propositions (1) and (2) have been removed.

(8) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ and $\mathrm{rng}\, f \subseteq Z$ holds f is a function from X into Z.

(9) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ and $Y \subseteq Z$ holds f is a function from X into Z.

In this article we present several logical schemes. The scheme *FuncEx1* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the following condition is met:

- For every x such that $x \in \mathcal{A}$ there exists y such that $y \in \mathcal{B}$ and $\mathcal{P}[x, y]$.

The scheme *Lambda1* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every x such that $x \in \mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

provided the following condition is met:

- For every x such that $x \in \mathcal{A}$ holds $\mathcal{F}(x) \in \mathcal{B}$.

Let us consider X, Y. The functor Y^X yielding a set is defined as follows:

(Def. 2) $x \in Y^X$ iff there exists a function f such that $x = f$ and $\mathrm{dom}\, f = X$ and $\mathrm{rng}\, f \subseteq Y$.

We now state two propositions:

(11)[2] For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $f \in Y^X$.

(12) For every function f from X into X holds $f \in X^X$.

Let X be a set and let Y be a non empty set. One can verify that Y^X is non empty.

Let X be a set. Note that X^X is non empty.

The following propositions are true:

(14)[3] If $X \neq \emptyset$, then $\emptyset^X = \emptyset$.

(16)[4] Let f be a function from X into Y. Suppose $Y \neq \emptyset$ and for every y such that $y \in Y$ there exists x such that $x \in X$ and $y = f(x)$. Then $\mathrm{rng}\, f = Y$.

(17) For every function f from X into Y such that $y \in Y$ and $\mathrm{rng}\, f = Y$ there exists x such that $x \in X$ and $f(x) = y$.

(18) For all functions f_1, f_2 from X into Y such that for every x such that $x \in X$ holds $f_1(x) = f_2(x)$ holds $f_1 = f_2$.

(19) Let f be a function from X into Y and g be a function from Y into Z such that if $Y = \emptyset$, then $Z = \emptyset$ or $X = \emptyset$. Then $g \cdot f$ is a function from X into Z.

(20) Let f be a function from X into Y and g be a function from Y into Z. If $Y \neq \emptyset$ and $Z \neq \emptyset$ and $\mathrm{rng}\, f = Y$ and $\mathrm{rng}\, g = Z$, then $\mathrm{rng}(g \cdot f) = Z$.

(21) For every function f from X into Y and for every function g such that $Y \neq \emptyset$ and $x \in X$ holds $(g \cdot f)(x) = g(f(x))$.

(22) Let f be a function from X into Y. Suppose $Y \neq \emptyset$. Then $\mathrm{rng}\, f = Y$ if and only if for every Z such that $Z \neq \emptyset$ and for all functions g, h from Y into Z such that $g \cdot f = h \cdot f$ holds $g = h$.

(23) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $f \cdot \mathrm{id}_X = f$ and $\mathrm{id}_Y \cdot f = f$.

[2] The proposition (10) has been removed.
[3] The proposition (13) has been removed.
[4] The proposition (15) has been removed.

(24) For every function f from X into Y and for every function g from Y into X such that $f \cdot g = \mathrm{id}_Y$ holds $\mathrm{rng}\, f = Y$.

(25) Let f be a function from X into Y such that if $Y = \emptyset$, then $X = \emptyset$. Then f is one-to-one if and only if for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ and $f(x_1) = f(x_2)$ holds $x_1 = x_2$.

(26) Let f be a function from X into Y and g be a function from Y into Z. Suppose if $Z = \emptyset$, then $Y = \emptyset$ and if $Y = \emptyset$, then $X = \emptyset$ and $g \cdot f$ is one-to-one. Then f is one-to-one.

(27) Let f be a function from X into Y. Suppose $X \neq \emptyset$ and $Y \neq \emptyset$. Then f is one-to-one if and only if for every Z and for all functions g, h from Z into X such that $f \cdot g = f \cdot h$ holds $g = h$.

(28) Let f be a function from X into Y and g be a function from Y into Z. If $Z \neq \emptyset$ and $\mathrm{rng}(g \cdot f) = Z$ and g is one-to-one, then $\mathrm{rng}\, f = Y$.

(29) Let f be a function from X into Y and g be a function from Y into X. If $Y \neq \emptyset$ and $g \cdot f = \mathrm{id}_X$, then f is one-to-one and $\mathrm{rng}\, g = X$.

(30) Let f be a function from X into Y and g be a function from Y into Z. Suppose if $Z = \emptyset$, then $Y = \emptyset$ and $g \cdot f$ is one-to-one and $\mathrm{rng}\, f = Y$. Then f is one-to-one and g is one-to-one.

(31) For every function f from X into Y such that f is one-to-one and $\mathrm{rng}\, f = Y$ holds f^{-1} is a function from Y into X.

(32) For every function f from X into Y such that $Y \neq \emptyset$ and f is one-to-one and $x \in X$ holds $f^{-1}(f(x)) = x$.

(34)[5] Let f be a function from X into Y and g be a function from Y into X. Suppose $X \neq \emptyset$ and $Y \neq \emptyset$ and $\mathrm{rng}\, f = Y$ and f is one-to-one and for all y, x holds $y \in Y$ and $g(y) = x$ iff $x \in X$ and $f(x) = y$. Then $g = f^{-1}$.

(35) For every function f from X into Y such that $Y \neq \emptyset$ and $\mathrm{rng}\, f = Y$ and f is one-to-one holds $f^{-1} \cdot f = \mathrm{id}_X$ and $f \cdot f^{-1} = \mathrm{id}_Y$.

(36) Let f be a function from X into Y and g be a function from Y into X. If $X \neq \emptyset$ and $Y \neq \emptyset$ and $\mathrm{rng}\, f = Y$ and $g \cdot f = \mathrm{id}_X$ and f is one-to-one, then $g = f^{-1}$.

(37) Let f be a function from X into Y. Suppose $Y \neq \emptyset$ and there exists a function g from Y into X such that $g \cdot f = \mathrm{id}_X$. Then f is one-to-one.

(38) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ and $Z \subseteq X$ holds $f \restriction Z$ is a function from Z into Y.

(40)[6] For every function f from X into Y such that $X \subseteq Z$ holds $f \restriction Z = f$.

(41) For every function f from X into Y such that $Y \neq \emptyset$ and $x \in X$ and $f(x) \in Z$ holds $(Z \restriction f)(x) = f(x)$.

(43)[7] Let f be a function from X into Y. Suppose $Y \neq \emptyset$. Let given y. If there exists x such that $x \in X$ and $x \in P$ and $y = f(x)$, then $y \in f^{\circ}P$.

(44) For every function f from X into Y holds $f^{\circ}P \subseteq Y$.

Let us consider X, Y, let f be a function from X into Y, and let us consider P. Then $f^{\circ}P$ is a subset of Y.

Next we state three propositions:

(45) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $f^{\circ}X = \mathrm{rng}\, f$.

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[5] The proposition (33) has been removed.
[6] The proposition (39) has been removed.
[7] The proposition (42) has been removed.

(46) For every function f from X into Y such that $Y \neq \emptyset$ and for every x holds $x \in f^{-1}(Q)$ iff $x \in X$ and $f(x) \in Q$.

(47) For every partial function f from X to Y holds $f^{-1}(Q) \subseteq X$.

Let us consider X, Y, let f be a partial function from X to Y, and let us consider Q. Then $f^{-1}(Q)$ is a subset of X.

The following propositions are true:

(48) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $f^{-1}(Y) = X$.

(49) For every function f from X into Y holds for every y such that $y \in Y$ holds $f^{-1}(\{y\}) \neq \emptyset$ iff $\operatorname{rng} f = Y$.

(50) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ and $P \subseteq X$ holds $P \subseteq f^{-1}(f^{\circ}P)$.

(51) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $f^{-1}(f^{\circ}X) = X$.

(53)[8] Let f be a function from X into Y and g be a function from Y into Z. If if $Z = \emptyset$, then $Y = \emptyset$ and if $Y = \emptyset$, then $X = \emptyset$, then $f^{-1}(Q) \subseteq (g \cdot f)^{-1}(g^{\circ}Q)$.

(55)[9] For every function f such that $\operatorname{dom} f = \emptyset$ holds f is a function from $\emptyset$ into Y.

(59)[10] For every function f from $\emptyset$ into Y holds $f^{\circ}P = \emptyset$.

(60) For every function f from $\emptyset$ into Y holds $f^{-1}(Q) = \emptyset$.

(61) For every function f from $\{x\}$ into Y such that $Y \neq \emptyset$ holds $f(x) \in Y$.

(62) For every function f from $\{x\}$ into Y such that $Y \neq \emptyset$ holds $\operatorname{rng} f = \{f(x)\}$.

(64)[11] For every function f from $\{x\}$ into Y such that $Y \neq \emptyset$ holds $f^{\circ}P \subseteq \{f(x)\}$.

(65) For every function f from X into $\{y\}$ such that $x \in X$ holds $f(x) = y$.

(66) For all functions f_1, f_2 from X into $\{y\}$ holds $f_1 = f_2$.

Let us consider X and let f, g be functions from X into X. Then $g \cdot f$ is a function from X into X.

One can prove the following propositions:

(67) For every function f from X into X holds $\operatorname{dom} f = X$ and $\operatorname{rng} f \subseteq X$.

(70)[12] For every function f from X into X and for every function g such that $x \in X$ holds $(g \cdot f)(x) = g(f(x))$.

(73)[13] For all functions f, g from X into X such that $\operatorname{rng} f = X$ and $\operatorname{rng} g = X$ holds $\operatorname{rng}(g \cdot f) = X$.

(74) For every function f from X into X holds $f \cdot \mathrm{id}_X = f$ and $\mathrm{id}_X \cdot f = f$.

(75) For all functions f, g from X into X such that $g \cdot f = f$ and $\operatorname{rng} f = X$ holds $g = \mathrm{id}_X$.

(76) For all functions f, g from X into X such that $f \cdot g = f$ and f is one-to-one holds $g = \mathrm{id}_X$.

(77) Let f be a function from X into X. Then f is one-to-one if and only if for all x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ and $f(x_1) = f(x_2)$ holds $x_1 = x_2$.

[8] The proposition (52) has been removed.
[9] The proposition (54) has been removed.
[10] The propositions (56)–(58) have been removed.
[11] The proposition (63) has been removed.
[12] The propositions (68) and (69) have been removed.
[13] The propositions (71) and (72) have been removed.

(79)[14] For every function f from X into X holds $f^\circ X = \operatorname{rng} f$.

(82)[15] For every function f from X into X holds $f^{-1}(f^\circ X) = X$.

Let X, Y be sets and let f be a function from X into Y. We say that f is onto if and only if:

(Def. 3) $\operatorname{rng} f = Y$.

Let us consider X, Y and let f be a function from X into Y. We say that f is bijective if and only if:

(Def. 4) f is one-to-one and onto.

Let X, Y be sets. Observe that every function from X into Y which is bijective is also one-to-one and onto and every function from X into Y which is one-to-one and onto is also bijective.

Let us consider X. One can check that there exists a function from X into X which is bijective.

Let us consider X. A permutation of X is a bijective function from X into X.

We now state two propositions:

(83) For every function f from X into X such that f is one-to-one and $\operatorname{rng} f = X$ holds f is a permutation of X.

(85)[16] Let f be a function from X into X. Suppose f is one-to-one. Let given x_1, x_2. If $x_1 \in X$ and $x_2 \in X$ and $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Let us consider X and let f, g be permutations of X. Then $g \cdot f$ is a permutation of X.

Let us consider X. Observe that every function from X into X which is reflexive and total is also bijective.

Let us consider X and let f be a permutation of X. Then f^{-1} is a permutation of X.

Next we state four propositions:

(86) For all permutations f, g of X such that $g \cdot f = g$ holds $f = \mathrm{id}_X$.

(87) For all permutations f, g of X such that $g \cdot f = \mathrm{id}_X$ holds $g = f^{-1}$.

(88) For every permutation f of X holds $f^{-1} \cdot f = \mathrm{id}_X$ and $f \cdot f^{-1} = \mathrm{id}_X$.

(92)[17] For every permutation f of X such that $P \subseteq X$ holds $f^\circ f^{-1}(P) = P$ and $f^{-1}(f^\circ P) = P$.

In the sequel C, D, E denote non empty sets.

Let us consider X, D. Note that every partial function from X to D which is quasi total is also total.

Let us consider X, D, Z, let f be a function from X into D, and let g be a function from D into Z. Then $g \cdot f$ is a function from X into Z.

In the sequel c denotes an element of C and d denotes an element of D.

Let C be a non empty set, let D be a set, let f be a function from C into D, and let c be an element of C. Then $f(c)$ is an element of D.

Now we present two schemes. The scheme *FuncExD* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the following condition is met:

- For every element x of $\mathcal{A}$ there exists an element y of $\mathcal{B}$ such that $\mathcal{P}[x, y]$.

The scheme *LambdaD* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

There exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

for all values of the parameters.

We now state several propositions:

[14] The proposition (78) has been removed.
[15] The propositions (80) and (81) have been removed.
[16] The proposition (84) has been removed.
[17] The propositions (89)–(91) have been removed.

(113)[18] For all functions f_1, f_2 from X into Y such that for every element x of X holds $f_1(x) = f_2(x)$ holds $f_1 = f_2$.

(115)[19] Let P be a set, f be a function from X into Y, and given y. If $y \in f^{\circ}P$, then there exists x such that $x \in X$ and $x \in P$ and $y = f(x)$.

(116) For every function f from X into Y and for every y such that $y \in f^{\circ}P$ there exists an element c of X such that $c \in P$ and $y = f(c)$.

(118)[20] For all functions f_1, f_2 from $[:X, Y:]$ into Z such that for all x, y such that $x \in X$ and $y \in Y$ holds $f_1(\langle x, y\rangle) = f_2(\langle x, y\rangle)$ holds $f_1 = f_2$.

(119) For every function f from $[:X, Y:]$ into Z such that $x \in X$ and $y \in Y$ and $Z \neq \emptyset$ holds $f(\langle x, y\rangle) \in Z$.

Now we present two schemes. The scheme *FuncEx2* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a ternary predicate $\mathcal{P}$, and states that:

There exists a function f from $[:\mathcal{A}, \mathcal{B}:]$ into $\mathcal{C}$ such that for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ holds $\mathcal{P}[x, y, f(\langle x, y\rangle)]$

provided the parameters satisfy the following condition:

- For all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ there exists z such that $z \in \mathcal{C}$ and $\mathcal{P}[x, y, z]$.

The scheme *Lambda2* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a binary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function f from $[:\mathcal{A}, \mathcal{B}:]$ into $\mathcal{C}$ such that for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ holds $f(\langle x, y\rangle) = \mathcal{F}(x, y)$

provided the following condition is met:

- For all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ holds $\mathcal{F}(x, y) \in \mathcal{C}$.

The following proposition is true

(120) For all functions f_1, f_2 from $[:C, D:]$ into E such that for all c, d holds $f_1(\langle c, d\rangle) = f_2(\langle c, d\rangle)$ holds $f_1 = f_2$.

Now we present two schemes. The scheme *FuncEx2D* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a ternary predicate $\mathcal{P}$, and states that:

There exists a function f from $[:\mathcal{A}, \mathcal{B}:]$ into $\mathcal{C}$ such that for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $\mathcal{P}[x, y, f(\langle x, y\rangle)]$

provided the parameters meet the following requirement:

- For every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ there exists an element z of $\mathcal{C}$ such that $\mathcal{P}[x, y, z]$.

The scheme *Lambda2D* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and states that:

There exists a function f from $[:\mathcal{A}, \mathcal{B}:]$ into $\mathcal{C}$ such that for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $f(\langle x, y\rangle) = \mathcal{F}(x, y)$

for all values of the parameters.

2. Partial functions from a set to a set (from [2])

Next we state the proposition

(121) For every set f such that $f \in Y^X$ holds f is a function from X into Y.

The scheme *Lambda1C* deals with sets $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

There exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every x such that $x \in \mathcal{A}$ holds if $\mathcal{P}[x]$, then $f(x) = \mathcal{F}(x)$ and if not $\mathcal{P}[x]$, then $f(x) = \mathcal{G}(x)$

[18] The propositions (93)–(112) have been removed.
[19] The proposition (114) has been removed.
[20] The proposition (117) has been removed.

provided the following requirement is met:

- For every x such that $x \in \mathcal{A}$ holds if $\mathcal{P}[x]$, then $\mathcal{F}(x) \in \mathcal{B}$ and if not $\mathcal{P}[x]$, then $\mathcal{G}(x) \in \mathcal{B}$.

We now state a number of propositions:

(130)[21] For every partial function f from X to Y such that $\operatorname{dom} f = X$ holds f is a function from X into Y.

(131) For every partial function f from X to Y such that f is total holds f is a function from X into Y.

(132) Let f be a partial function from X to Y. Suppose if $Y = \emptyset$, then $X = \emptyset$ and f is a function from X into Y. Then f is total.

(133) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $f_{\restriction X \dot{\to} Y}$ is total.

(134) For every function f from X into X holds $f_{\restriction X \dot{\to} X}$ is total.

(136)[22] Let f be a partial function from X to Y such that if $Y = \emptyset$, then $X = \emptyset$. Then there exists a function g from X into Y such that for every x such that $x \in \operatorname{dom} f$ holds $g(x) = f(x)$.

(141)[23] $Y^X \subseteq X \dot{\to} Y$.

(142) For all functions f, g from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ and $f \approx g$ holds $f = g$.

(143) For all functions f, g from X into X such that $f \approx g$ holds $f = g$.

(145)[24] Let f be a partial function from X to Y and g be a function from X into Y such that if $Y = \emptyset$, then $X = \emptyset$. Then $f \approx g$ if and only if for every x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$.

(146) Let f be a partial function from X to X and g be a function from X into X. Then $f \approx g$ if and only if for every x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$.

(148)[25] For every partial function f from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ there exists a function g from X into Y such that $f \approx g$.

(149) For every partial function f from X to X there exists a function g from X into X such that $f \approx g$.

(151)[26] Let f, g be partial functions from X to Y and h be a function from X into Y. If if $Y = \emptyset$, then $X = \emptyset$ and $f \approx h$ and $g \approx h$, then $f \approx g$.

(152) Let f, g be partial functions from X to X and h be a function from X into X. If $f \approx h$ and $g \approx h$, then $f \approx g$.

(154)[27] Let f, g be partial functions from X to Y. Suppose if $Y = \emptyset$, then $X = \emptyset$ and $f \approx g$. Then there exists a function h from X into Y such that $f \approx h$ and $g \approx h$.

(155) Let f be a partial function from X to Y and g be a function from X into Y. If if $Y = \emptyset$, then $X = \emptyset$ and $f \approx g$, then $g \in \operatorname{TotFuncs} f$.

(156) For every partial function f from X to X and for every function g from X into X such that $f \approx g$ holds $g \in \operatorname{TotFuncs} f$.

(158)[28] Let f be a partial function from X to Y and g be a set. If $g \in \operatorname{TotFuncs} f$, then g is a function from X into Y.

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[21] The propositions (122)–(129) have been removed.
[22] The proposition (135) has been removed.
[23] The propositions (137)–(140) have been removed.
[24] The proposition (144) has been removed.
[25] The proposition (147) has been removed.
[26] The proposition (150) has been removed.
[27] The proposition (153) has been removed.
[28] The proposition (157) has been removed.

(159) For every partial function f from X to Y holds $\mathrm{TotFuncs}\, f \subseteq Y^X$.

(160) $\mathrm{TotFuncs}(\emptyset_{\restriction X \dot{\to} Y}) = Y^X$.

(161) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $\mathrm{TotFuncs}(f_{\restriction X \dot{\to} Y}) = \{f\}$.

(162) For every function f from X into X holds $\mathrm{TotFuncs}(f_{\restriction X \dot{\to} X}) = \{f\}$.

(164)[29] For every partial function f from X to $\{y\}$ and for every function g from X into $\{y\}$ holds $\mathrm{TotFuncs}\, f = \{g\}$.

(165) For all partial functions f, g from X to Y such that $g \subseteq f$ holds $\mathrm{TotFuncs}\, f \subseteq \mathrm{TotFuncs}\, g$.

(166) For all partial functions f, g from X to Y such that $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f$ and $\mathrm{TotFuncs}\, f \subseteq \mathrm{TotFuncs}\, g$ holds $g \subseteq f$.

(167) For all partial functions f, g from X to Y such that $\mathrm{TotFuncs}\, f \subseteq \mathrm{TotFuncs}\, g$ and for every y holds $Y \neq \{y\}$ holds $g \subseteq f$.

(168) For all partial functions f, g from X to Y such that for every y holds $Y \neq \{y\}$ and $\mathrm{TotFuncs}\, f = \mathrm{TotFuncs}\, g$ holds $f = g$.

Let A, B be non empty sets. Note that every function from A into B is non empty.

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[29] The proposition (163) has been removed.