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Free Universal Algebra Construction

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Summary. A construction of the free universal algebra with fixed signature and a given set of generators.

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The articles [15], [18], [16], [19], [7], [9], [4], [8], [14], [10], [1], [2], [3], [17], [13], [6], [11], [5], and [12] provide the notation and terminology for this paper.

1. PRELIMINARIES

In this paper n denotes a natural number.

Let I_1 be a set. We say that I_1 is missing $\mathbb{N}$ if and only if:

(Def. 1) I_1 misses $\mathbb{N}$.

One can check that there exists a set which is non empty and missing $\mathbb{N}$.

Let I_1 be a finite sequence. We say that I_1 has zero if and only if:

(Def. 2) $0 \in \operatorname{rng} I_1$.

We introduce I_1 is without zero as an antonym of I_1 has zero.

Let us note that there exists a finite sequence of elements of $\mathbb{N}$ which is non empty and has zero and there exists a finite sequence of elements of $\mathbb{N}$ which is non empty and without zero.

2. FREE UNIVERSAL ALGEBRA — GENERAL NOTIONS

Let U_1 be a universal algebra and let n be a natural number. Let us assume that $n \in \operatorname{dom}$(the characteristic of U_1). The functor $\operatorname{oper}(n, U_1)$ yields an operation of U_1 and is defined by:

(Def. 4)[1] $\operatorname{oper}(n, U_1) =$ (the characteristic of $U_1)(n)$.

Let U_0 be a universal algebra. A subset of U_0 is called a generator set of U_0 if:

(Def. 5) The carrier of $\mathrm{Gen}^{\mathrm{UA}}(\mathrm{it}) =$ the carrier of U_0.

Let U_0 be a universal algebra and let I_1 be a generator set of U_0. We say that I_1 is free if and only if the condition (Def. 6) is satisfied.

[1] The definition (Def. 3) has been removed.

(Def. 6) Let U_1 be a universal algebra. Suppose U_0 and U_1 are similar. Let f be a function from I_1 into the carrier of U_1. Then there exists a function h from U_0 into U_1 such that h is a homomorphism of U_0 into U_1 and $h \restriction I_1 = f$.

Let I_1 be a universal algebra. We say that I_1 is free if and only if:

(Def. 7) There exists a generator set of I_1 which is free.

Let us observe that there exists a universal algebra which is free and strict.
Let U_0 be a free universal algebra. Observe that there exists a generator set of U_0 which is free.
We now state the proposition

(1) Let U_0 be a strict universal algebra and A be a subset of U_0. Then A is a generator set of U_0 if and only if $\mathrm{Gen}^{\mathrm{UA}}(A) = U_0$.

3. Construction of Decorated Tree Structure for Free Universal Algebra

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let X be a set. The functor $\mathrm{REL}(f,X)$ yielding a relation between $\mathrm{dom}\, f \cup X$ and $(\mathrm{dom}\, f \cup X)^*$ is defined by:

(Def. 8) For every element a of $\mathrm{dom}\, f \cup X$ and for every element b of $(\mathrm{dom}\, f \cup X)^*$ holds $\langle a, b\rangle \in \mathrm{REL}(f,X)$ iff $a \in \mathrm{dom}\, f$ and $f(a) = \mathrm{len}\, b$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let X be a set. The functor $\mathrm{DTConUA}(f,X)$ yielding a strict tree construction structure is defined as follows:

(Def. 9) $\mathrm{DTConUA}(f,X) = \langle \mathrm{dom}\, f \cup X, \mathrm{REL}(f,X)\rangle$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let X be a set. One can check that $\mathrm{DTConUA}(f,X)$ is non empty.
The following propositions are true:

(2) Let f be a non empty finite sequence of elements of $\mathbb{N}$ and X be a set. Then the terminals of $\mathrm{DTConUA}(f,X) \subseteq X$ and the nonterminals of $\mathrm{DTConUA}(f,X) = \mathrm{dom}\, f$.

(3) Let f be a non empty finite sequence of elements of $\mathbb{N}$ and X be a missing $\mathbb{N}$ set. Then the terminals of $\mathrm{DTConUA}(f,X) = X$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let X be a set. Note that $\mathrm{DTConUA}(f,X)$ has nonterminals.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let X be a set. Note that $\mathrm{DTConUA}(f,X)$ has nonterminals and useful nonterminals.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a missing $\mathbb{N}$ non empty set. Note that $\mathrm{DTConUA}(f,D)$ has terminals, nonterminals, and useful nonterminals.

Let f be a non empty finite sequence of elements of $\mathbb{N}$, let X be a set, and let n be a natural number. Let us assume that $n \in \mathrm{dom}\, f$. The functor $\mathrm{Sym}(n,f,X)$ yields a symbol of $\mathrm{DTConUA}(f,X)$ and is defined as follows:

(Def. 10) $\mathrm{Sym}(n,f,X) = n$.

4. Construction of Free Universal Algebra for Non-empty Set of Generators and Given Signature

Let f be a non empty finite sequence of elements of $\mathbb{N}$ **qua** non empty set, let D be a missing $\mathbb{N}$ non empty set, and let n be a natural number. Let us assume that $n \in \mathrm{dom}\, f$. The functor $\mathrm{FreeOpNSG}(n,f,D)$ yielding a homogeneous quasi total non empty partial function from $\mathrm{TS}(\mathrm{DTConUA}(f,D))^*$ to $\mathrm{TS}(\mathrm{DTConUA}(f,D))$ is defined by the conditions (Def. 11).

(Def. 11)(i) $\mathrm{dom}\,\mathrm{FreeOpNSG}(n,f,D) = \mathrm{TS}(\mathrm{DTConUA}(f,D))^{f_n}$, and

(ii) for every finite sequence p of elements of $\mathrm{TS}(\mathrm{DTConUA}(f,D))$ such that $p \in \mathrm{dom}\,\mathrm{FreeOpNSG}(n,f,D)$ holds $(\mathrm{FreeOpNSG}(n,f,D))(p) = \mathrm{Sym}(n,f,D)\text{-tree}(p)$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a missing $\mathbb{N}$ non empty set. The functor FreeOpSeqNSG(f,D) yields a finite sequence of operational functions of TS(DTConUA(f,D)) and is defined by:

(Def. 12) len FreeOpSeqNSG$(f,D) = \operatorname{len} f$ and for every n such that $n \in \operatorname{dom}$ FreeOpSeqNSG(f,D) holds (FreeOpSeqNSG$(f,D))(n) =$ FreeOpNSG(n,f,D).

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a missing $\mathbb{N}$ non empty set. The functor FreeUnivAlgNSG(f,D) yields a strict universal algebra and is defined as follows:

(Def. 13) FreeUnivAlgNSG$(f,D) = \langle$TS(DTConUA(f,D)), FreeOpSeqNSG$(f,D)\rangle$.

The following proposition is true

(4) For every non empty finite sequence f of elements of $\mathbb{N}$ and for every missing $\mathbb{N}$ non empty set D holds signature FreeUnivAlgNSG$(f,D) = f$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a non empty missing $\mathbb{N}$ set. The functor FreeGenSetNSG(f,D) yielding a subset of FreeUnivAlgNSG(f,D) is defined as follows:

(Def. 14) FreeGenSetNSG$(f,D) = \{$the root tree of s; s ranges over symbols of DTConUA(f,D): $s \in$ the terminals of DTConUA$(f,D)\}$.

One can prove the following proposition

(5) Let f be a non empty finite sequence of elements of $\mathbb{N}$ and D be a non empty missing $\mathbb{N}$ set. Then FreeGenSetNSG(f,D) is non empty.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ **qua** non empty set and let D be a non empty missing $\mathbb{N}$ set. Then FreeGenSetNSG(f,D) is a generator set of FreeUnivAlgNSG(f,D).

Let f be a non empty finite sequence of elements of $\mathbb{N}$, let D be a non empty missing $\mathbb{N}$ set, let C be a non empty set, let s be a symbol of DTConUA(f,D), and let F be a function from FreeGenSetNSG(f,D) into C. Let us assume that $s \in$ the terminals of DTConUA(f,D). The functor $\pi_s F$ yields an element of C and is defined as follows:

(Def. 15) $\pi_s F = F$(the root tree of s).

Let f be a non empty finite sequence of elements of $\mathbb{N}$, let D be a set, and let s be a symbol of DTConUA(f,D). Let us assume that there exists a finite sequence p such that $s \Rightarrow p$. The functor ${}^{@}s$ yields a natural number and is defined by:

(Def. 16) ${}^{@}s = s$.

The following proposition is true

(6) For every non empty finite sequence f of elements of $\mathbb{N}$ and for every non empty missing $\mathbb{N}$ set D holds FreeGenSetNSG(f,D) is free.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a non empty missing $\mathbb{N}$ set. Note that FreeUnivAlgNSG(f,D) is free.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a non empty missing $\mathbb{N}$ set. Then FreeGenSetNSG(f,D) is a free generator set of FreeUnivAlgNSG(f,D).

5. Construction of Free Universal Algebra and Set of Generators

Let f be a non empty finite sequence of elements of $\mathbb{N}$ **qua** non empty set with zero, let D be a missing $\mathbb{N}$ set, and let n be a natural number. Let us assume that $n \in \operatorname{dom} f$. The functor FreeOpZAO$(n,f,D)$ yields a homogeneous quasi total non empty partial function from TS(DTConUA$(f,D))^*$ to TS(DTConUA(f,D)) and is defined by the conditions (Def. 17).

(Def. 17)(i) $\operatorname{dom}\mathrm{FreeOpZAO}(n,f,D)=\mathrm{TS}(\mathrm{DTConUA}(f,D))^{f_n}$, and

(ii) for every finite sequence p of elements of $\mathrm{TS}(\mathrm{DTConUA}(f,D))$ such that $p \in \operatorname{dom}\mathrm{FreeOpZAO}(n,f,D)$ holds $(\mathrm{FreeOpZAO}(n,f,D))(p)=\mathrm{Sym}(n,f,D)\text{-tree}(p)$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. The functor $\mathrm{FreeOpSeqZAO}(f,D)$ yields a finite sequence of operational functions of $\mathrm{TS}(\mathrm{DTConUA}(f,D))$ and is defined by:

(Def. 18) $\operatorname{len}\mathrm{FreeOpSeqZAO}(f,D)=\operatorname{len}f$ and for every n such that $n \in \operatorname{dom}\mathrm{FreeOpSeqZAO}(f,D)$ holds $(\mathrm{FreeOpSeqZAO}(f,D))(n)=\mathrm{FreeOpZAO}(n,f,D)$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. The functor $\mathrm{FreeUnivAlgZAO}(f,D)$ yields a strict universal algebra and is defined by:

(Def. 19) $\mathrm{FreeUnivAlgZAO}(f,D)=\langle\mathrm{TS}(\mathrm{DTConUA}(f,D)),\mathrm{FreeOpSeqZAO}(f,D)\rangle$.

Next we state three propositions:

(7) For every non empty finite sequence f of elements of $\mathbb{N}$ with zero and for every missing $\mathbb{N}$ set D holds $\operatorname{signature}\mathrm{FreeUnivAlgZAO}(f,D)=f$.

(8) Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and D be a missing $\mathbb{N}$ set. Then $\mathrm{FreeUnivAlgZAO}(f,D)$ has constants.

(9) For every non empty finite sequence f of elements of $\mathbb{N}$ with zero and for every missing $\mathbb{N}$ set D holds $\mathrm{Constants}(\mathrm{FreeUnivAlgZAO}(f,D)) \neq \emptyset$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. The functor $\mathrm{FreeGenSetZAO}(f,D)$ yielding a subset of $\mathrm{FreeUnivAlgZAO}(f,D)$ is defined by:

(Def. 20) $\mathrm{FreeGenSetZAO}(f,D)=\{$the root tree of s; s ranges over symbols of $\mathrm{DTConUA}(f,D)$: $s \in$ the terminals of $\mathrm{DTConUA}(f,D)\}$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ **qua** non empty set with zero and let D be a missing $\mathbb{N}$ set. Then $\mathrm{FreeGenSetZAO}(f,D)$ is a generator set of $\mathrm{FreeUnivAlgZAO}(f,D)$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero, let D be a missing $\mathbb{N}$ set, let C be a non empty set, let s be a symbol of $\mathrm{DTConUA}(f,D)$, and let F be a function from $\mathrm{FreeGenSetZAO}(f,D)$ into C. Let us assume that $s \in$ the terminals of $\mathrm{DTConUA}(f,D)$. The functor $\pi_s F$ yields an element of C and is defined by:

(Def. 21) $\pi_s F = F(\text{the root tree of } s)$.

We now state the proposition

(10) For every non empty finite sequence f of elements of $\mathbb{N}$ with zero and for every missing $\mathbb{N}$ set D holds $\mathrm{FreeGenSetZAO}(f,D)$ is free.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. One can check that $\mathrm{FreeUnivAlgZAO}(f,D)$ is free.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. Then $\mathrm{FreeGenSetZAO}(f,D)$ is a free generator set of $\mathrm{FreeUnivAlgZAO}(f,D)$.

Let us observe that there exists a universal algebra which is strict and free and has constants.

REFERENCES

[1] Grzegorz Bancerek. König's Lemma. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/trees_2.html.

[2] Grzegorz Bancerek. Sets and functions of trees and joining operations of trees. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/trees_3.html.

[3] Grzegorz Bancerek. Joining of decorated trees. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/trees_4.html.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[5] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/dtconstr.html.

[6] Ewa Burakowska. Subalgebras of the universal algebra. Lattices of subalgebras. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/unialg_2.html.

[7] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[8] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[9] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[10] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[11] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar — part I. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/lang1.html.

[12] Małgorzata Korolkiewicz. Homomorphisms of algebras. Quotient universal algebra. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/alg_1.html.

[13] Jarosław Kotowicz, Beata Madras, and Małgorzata Korolkiewicz. Basic notation of universal algebra. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/unialg_1.html.

[14] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funcop_1.html.

[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[16] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[17] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[18] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[19] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

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