JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Function Domains and Frænkel Operator

Andrzej Trybulec
Warsaw University
Białystok

Summary. We deal with a non–empty set of functions and a non–empty set of functions from a set A to a non–empty set B. In the case when B is a non–empty set, B^A is redefined. It yields a non–empty set of functions from A to B. An element of such a set is redefined as a function from A to B. Some theorems concerning these concepts are proved, as well as a number of schemes dealing with infinity and the Axiom of Choice. The article contains a number of schemes allowing for simple logical transformations related to terms constructed with the Frænkel Operator.

MML Identifier: FRAENKEL.

WWW: http://mizar.org/JFM/Vol2/fraenkel.html

The articles [6], [3], [8], [9], [4], [7], [1], [2], and [5] provide the notation and terminology for this paper.

In this paper B is a non empty set and A, C, X are sets.

In this article we present several logical schemes. The scheme *Fraenkel5'* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(v'); v'$ ranges over elements of $\mathcal{A} : \mathcal{P}[v']\} \subseteq \{\mathcal{F}(u'); u'$ ranges over elements of $\mathcal{A} : \mathcal{Q}[u']\}$

provided the following condition is satisfied:

- For every element v of $\mathcal{A}$ such that $\mathcal{P}[v]$ holds $\mathcal{Q}[v]$.

The scheme *Fraenkel5"* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(u_1, v_1); u_1$ ranges over elements of $\mathcal{A}, v_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_1, v_1]\} \subseteq \{\mathcal{F}(u_2, v_2); u_2$ ranges over elements of $\mathcal{A}, v_2$ ranges over elements of $\mathcal{B} : \mathcal{Q}[u_2, v_2]\}$

provided the parameters meet the following requirement:

- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ such that $\mathcal{P}[u, v]$ holds $\mathcal{Q}[u, v]$.

The scheme *Fraenkel6'* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(v_1); v_1$ ranges over elements of $\mathcal{A} : \mathcal{P}[v_1]\} = \{\mathcal{F}(v_2); v_2$ ranges over elements of $\mathcal{A} : \mathcal{Q}[v_2]\}$

provided the following condition is satisfied:

- For every element v of $\mathcal{A}$ holds $\mathcal{P}[v]$ iff $\mathcal{Q}[v]$.

The scheme *Fraenkel6"* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(u_1, v_1); u_1$ ranges over elements of $\mathcal{A}, v_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_1, v_1]\} = \{\mathcal{F}(u_2, v_2); u_2$ ranges over elements of $\mathcal{A}, v_2$ ranges over elements of $\mathcal{B} : \mathcal{Q}[u_2, v_2]\}$

provided the following condition is met:

- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{P}[u, v]$ iff $\mathcal{Q}[u, v]$.

The scheme *FraenkelF'* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(v_1); v_1$ ranges over elements of $\mathcal{A} : \mathcal{P}[v_1]\} = \{\mathcal{G}(v_2); v_2$ ranges over elements of $\mathcal{A} : \mathcal{P}[v_2]\}$

provided the parameters have the following property:

- For every element v of $\mathcal{A}$ holds $\mathcal{F}(v) = \mathcal{G}(v)$.

The scheme *FraenkelF'R* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(v_1); v_1$ ranges over elements of $\mathcal{A} : \mathcal{P}[v_1]\} = \{\mathcal{G}(v_2); v_2$ ranges over elements of $\mathcal{A} : \mathcal{P}[v_2]\}$

provided the following condition is met:

- For every element v of $\mathcal{A}$ such that $\mathcal{P}[v]$ holds $\mathcal{F}(v) = \mathcal{G}(v)$.

The scheme *FraenkelF"* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, a binary functor $\mathcal{G}$ yielding a set, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(u_1, v_1); u_1$ ranges over elements of $\mathcal{A}, v_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_1, v_1]\} = \{\mathcal{G}(u_2, v_2); u_2$ ranges over elements of $\mathcal{A}, v_2$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_2, v_2]\}$

provided the following condition is satisfied:

- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{F}(u, v) = \mathcal{G}(u, v)$.

The scheme *FraenkelF6"* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(u_1, v_1); u_1$ ranges over elements of $\mathcal{A}, v_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_1, v_1]\} = \{\mathcal{F}(v_2, u_2); u_2$ ranges over elements of $\mathcal{A}, v_2$ ranges over elements of $\mathcal{B} : \mathcal{Q}[u_2, v_2]\}$

provided the parameters have the following properties:

- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{P}[u, v]$ iff $\mathcal{Q}[u, v]$, and
- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{F}(u, v) = \mathcal{F}(v, u)$.

The following propositions are true:

(3)[1] Let A, B be non empty sets, F, G be functions from A into B, and X be a set. If $F{\restriction}X = G{\restriction}X$, then for every element x of A such that $x \in X$ holds $F(x) = G(x)$.

(5)[2] For all sets A, B holds $B^A \subseteq 2^{[:A, B:]}$.

(6) For all sets X, Y such that $Y^X \neq \emptyset$ and $X \subseteq A$ and $Y \subseteq B$ holds every element of Y^X is a partial function from A to B.

Now we present a number of schemes. The scheme *RelevantArgs* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a set $\mathcal{C}$, a function $\mathcal{D}$ from $\mathcal{A}$ into $\mathcal{B}$, a function $\mathcal{E}$ from $\mathcal{A}$ into $\mathcal{B}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{D}(u'); u'$ ranges over elements of $\mathcal{A} : \mathcal{P}[u'] \wedge u' \in \mathcal{C}\} = \{\mathcal{E}(v'); v'$ ranges over elements of $\mathcal{A} : \mathcal{Q}[v'] \wedge v' \in \mathcal{C}\}$

provided the parameters meet the following requirements:

- $\mathcal{D}{\restriction}\mathcal{C} = \mathcal{E}{\restriction}\mathcal{C}$, and
- For every element u of $\mathcal{A}$ such that $u \in \mathcal{C}$ holds $\mathcal{P}[u]$ iff $\mathcal{Q}[u]$.

The scheme *Fr Set0* deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

$\{x; x$ ranges over elements of $\mathcal{A} : \mathcal{P}[x]\} \subseteq \mathcal{A}$

for all values of the parameters.

The scheme *Gen1"* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

For every element s of $\mathcal{A}$ and for every element t of $\mathcal{B}$ such that $\mathcal{P}[s, t]$ holds $\mathcal{Q}[\mathcal{F}(s, t)]$

provided the parameters satisfy the following condition:

- For every set s_1 such that $s_1 \in \{\mathcal{F}(s_2, t_1); s_2$ ranges over elements of $\mathcal{A}, t_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[s_2, t_1]\}$ holds $\mathcal{Q}[s_1]$.

The scheme *Gen1"A* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

[1] The propositions (1) and (2) have been removed.

[2] The proposition (4) has been removed.

For every set s_1 such that $s_1 \in \{\mathcal{F}(s_2,t_1); s_2$ ranges over elements of $\mathcal{A}, t_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[s_2,t_1]\}$ holds $\mathcal{Q}[s_1]$
provided the following condition is met:

- For every element s of $\mathcal{A}$ and for every element t of $\mathcal{B}$ such that $\mathcal{P}[s,t]$ holds $\mathcal{Q}[\mathcal{F}(s,t)]$.

The scheme *Gen2"* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a non empty set $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

$\{s_1; s_1$ ranges over elements of $\mathcal{C} : s_1 \in \{\mathcal{F}(s_2,t_1); s_2$ ranges over elements of $\mathcal{A}, t_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[s_2,t_1]\} \wedge \mathcal{Q}[s_1]\} = \{\mathcal{F}(s_3,t_2); s_3$ ranges over elements of $\mathcal{A}, t_2$ ranges over elements of $\mathcal{B} : \mathcal{P}[s_3,t_2] \wedge \mathcal{Q}[\mathcal{F}(s_3,t_2)]\}$

for all values of the parameters.

The scheme *Gen3'* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(s); s$ ranges over elements of $\mathcal{A} : s \in \{s_2; s_2$ ranges over elements of $\mathcal{A} : \mathcal{Q}[s_2]\} \wedge \mathcal{P}[s]\} = \{\mathcal{F}(s_3); s_3$ ranges over elements of $\mathcal{A} : \mathcal{Q}[s_3] \wedge \mathcal{P}[s_3]\}$

for all values of the parameters.

The scheme *Gen3"* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(s,t); s$ ranges over elements of $\mathcal{A}, t$ ranges over elements of $\mathcal{B} : s \in \{s_2; s_2$ ranges over elements of $\mathcal{A} : \mathcal{Q}[s_2]\} \wedge \mathcal{P}[s,t]\} = \{\mathcal{F}(s_3,t_2); s_3$ ranges over elements of $\mathcal{A}, t_2$ ranges over elements of $\mathcal{B} : \mathcal{Q}[s_3] \wedge \mathcal{P}[s_3,t_2]\}$

for all values of the parameters.

The scheme *Gen4"* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(s,t); s$ ranges over elements of $\mathcal{A}, t$ ranges over elements of $\mathcal{B} : \mathcal{P}[s,t]\} \subseteq \{\mathcal{F}(s_2,t_1); s_2$ ranges over elements of $\mathcal{A}, t_1$ ranges over elements of $\mathcal{B} : \mathcal{Q}[s_2,t_1]\}$

provided the parameters meet the following condition:

- Let s be an element of $\mathcal{A}$ and t be an element of $\mathcal{B}$. If $\mathcal{P}[s,t]$, then there exists an element s' of $\mathcal{A}$ such that $\mathcal{Q}[s',t]$ and $\mathcal{F}(s,t) = \mathcal{F}(s',t)$.

The scheme *FrSet 1* deals with a non empty set $\mathcal{A}$, a set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(y); y$ ranges over elements of $\mathcal{A} : \mathcal{F}(y) \in \mathcal{B} \wedge \mathcal{P}[y]\} \subseteq \mathcal{B}$

for all values of the parameters.

The scheme *FrSet 2* deals with a non empty set $\mathcal{A}$, a set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(y); y$ ranges over elements of $\mathcal{A} : \mathcal{P}[y] \wedge \mathcal{F}(y) \notin \mathcal{B}\}$ misses $\mathcal{B}$

for all values of the parameters.

The scheme *FrEqua1* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, an element $\mathcal{C}$ of $\mathcal{B}$, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(s,t); s$ ranges over elements of $\mathcal{A}, t$ ranges over elements of $\mathcal{B} : \mathcal{Q}[s,t]\} = \{\mathcal{F}(s',\mathcal{C}); s'$ ranges over elements of $\mathcal{A} : \mathcal{P}[s',\mathcal{C}]\}$

provided the following condition is satisfied:

- For every element s of $\mathcal{A}$ and for every element t of $\mathcal{B}$ holds $\mathcal{Q}[s,t]$ iff $t = \mathcal{C}$ and $\mathcal{P}[s,t]$.

The scheme *FrEqua2* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding a set, an element $\mathcal{C}$ of $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(s,t); s$ ranges over elements of $\mathcal{A}, t$ ranges over elements of $\mathcal{B} : t = \mathcal{C} \wedge \mathcal{P}[s,t]\} = \{\mathcal{F}(s',\mathcal{C}); s'$ ranges over elements of $\mathcal{A} : \mathcal{P}[s',\mathcal{C}]\}$

for all values of the parameters.

Let I_1 be a set. We say that I_1 is functional if and only if:

(Def. 1) For every set x such that $x \in I_1$ holds x is a function.

One can verify that there exists a set which is non empty and functional.

Let P be a functional set. Observe that every element of P is function-like and relation-like.

The following proposition is true

(8)[3] For every function f holds $\{f\}$ is functional.

Let A, B be sets. Observe that B^A is functional.

Let A, B be sets. A functional non empty set is said to be a non empty set of functions from A to B if:

(Def. 2) Every element of it is a function from A into B.

We now state two propositions:

(10)[4] For every function f from A into C holds $\{f\}$ is a non empty set of functions from A to C.

(11) B^A is a non empty set of functions from A to B.

Let A be a set and let B be a non empty set. Then B^A is a non empty set of functions from A to B. Let F be a non empty set of functions from A to B. We see that the element of F is a function from A into B.

In the sequel p_1 is an element of B^A.

We now state two propositions:

(14)[5] Let X, Y be sets. Suppose $Y^X \neq \emptyset$ and $X \subseteq A$ and $Y \subseteq B$. Let f be an element of Y^X. Then there exists an element p_1 of B^A such that $p_1 \restriction X = f$.

(15) For every set X and for every p_1 holds $p_1 \restriction X = p_1 \restriction (A \cap X)$.

Now we present four schemes. The scheme *FraenkelFin* deals with a non empty set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\{\mathcal{F}(w); w$ ranges over elements of $\mathcal{A} : w \in \mathcal{B}\}$ is finite

provided the parameters meet the following requirement:

- $\mathcal{B}$ is finite.

The scheme *CartFin* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, sets $\mathcal{C}$, $\mathcal{D}$, and a binary functor $\mathcal{F}$ yielding a set, and states that:

$\{\mathcal{F}(u', v'); u'$ ranges over elements of $\mathcal{A}, v'$ ranges over elements of $\mathcal{B} : u' \in \mathcal{C} \wedge v' \in \mathcal{D}\}$ is finite

provided the parameters meet the following requirements:

- $\mathcal{C}$ is finite, and
- $\mathcal{D}$ is finite.

The scheme *Finiteness* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathrm{Fin}\,\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

Let x be an element of $\mathcal{A}$. Suppose $x \in \mathcal{B}$. Then there exists an element y of $\mathcal{A}$ such that $y \in \mathcal{B}$ and $\mathcal{P}[y, x]$ and for every element z of $\mathcal{A}$ such that $z \in \mathcal{B}$ and $\mathcal{P}[z, y]$ holds $\mathcal{P}[y, z]$

provided the following conditions are satisfied:

- For every element x of $\mathcal{A}$ holds $\mathcal{P}[x, x]$, and
- For all elements x, y, z of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

The scheme *Fin Im* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathrm{Fin}\,\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

There exists an element c_1 of $\mathrm{Fin}\,\mathcal{A}$ such that for every element t of $\mathcal{A}$ holds $t \in c_1$ if and only if there exists an element t' of $\mathcal{B}$ such that $t' \in \mathcal{C}$ and $t = \mathcal{F}(t')$ and $\mathcal{P}[t, t']$

for all values of the parameters.

The following proposition is true

(16) For all sets A, B such that A is finite and B is finite holds B^A is finite.

[3] The proposition (7) has been removed.

[4] The proposition (9) has been removed.

[5] The propositions (12) and (13) have been removed.

Now we present three schemes. The scheme *ImFin* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a set $\mathcal{C}$, a set $\mathcal{D}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\{\mathcal{F}(p'_1); p'_1$ ranges over elements of $\mathcal{B}^{\mathcal{A}}$: $p'_1{}^{\circ}\mathcal{C} \subseteq \mathcal{D}\}$ is finite

provided the parameters meet the following conditions:

- $\mathcal{C}$ is finite,
- $\mathcal{D}$ is finite, and
- For all elements p_1, p_2 of $\mathcal{B}^{\mathcal{A}}$ such that $p_1 \restriction \mathcal{C} = p_2 \restriction \mathcal{C}$ holds $\mathcal{F}(p_1) = \mathcal{F}(p_2)$.

The scheme *FunctChoice* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathrm{Fin}\,\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a function f_1 from $\mathcal{A}$ into $\mathcal{B}$ such that for every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ holds $\mathcal{P}[t, f_1(t)]$

provided the following requirement is met:

- For every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ there exists an element f_1 of $\mathcal{B}$ such that $\mathcal{P}[t, f_1]$.

The scheme *FuncsChoice* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathrm{Fin}\,\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

There exists an element f_1 of $\mathcal{B}^{\mathcal{A}}$ such that for every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ holds $\mathcal{P}[t, f_1(t)]$

provided the parameters meet the following requirement:

- For every element t of $\mathcal{A}$ such that $t \in \mathcal{C}$ there exists an element f_1 of $\mathcal{B}$ such that $\mathcal{P}[t, f_1]$.

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Received February 7, 1990

Published January 2, 2004