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Formal Topological Spaces

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Summary. This article is divided into two parts. In the first part, we prove some useful theorems on finite topological spaces. In the second part, in order to consider a family of neighborhoods to a point in a space, we extend the notion of finite topological space and define a new topological space, which we call formal topological space. We show the relation between formal topological space struct (`FMT_Space_Str`) and the `TopStruct` by giving a function (`NeighSp`). And the following notions are introduced in formal topological spaces: boundary, closure, interior, isolated point, connected point, open set and close set, then some basic facts concerning them are proved. We will discuss the relation between the formal topological space and the finite topological space in future work.

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The articles [6], [3], [7], [1], [2], [4], [5], and [8] provide the notation and terminology for this paper.

1. Some Useful Theorems on Finite Topological Spaces

In this paper F_1 is a non empty finite topology space and A is a subset of F_1.

Next we state several propositions:

(1) For every non empty finite topology space F_1 and for all subsets A, B of F_1 such that $A \subseteq B$ holds $A^i \subseteq B^i$.

(2) $A^\delta = A^b \cap (A^i)^c$.

(3) $A^\delta = A^b \setminus A^i$.

(4) If A^c is open, then A is closed.

(5) If A^c is closed, then A is open.

Let F_1 be a non empty finite topology space, let x be an element of F_1, let y be an element of F_1, and let A be a subset of F_1. The functor $P_1(x,y,A)$ yields an element of *Boolean* and is defined as follows:

(Def. 1) $$P_1(x,y,A) = \begin{cases} \mathit{true}, & \text{if } y \in U(x) \text{ and } y \in A, \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

Let F_1 be a non empty finite topology space, let x be an element of F_1, let y be an element of F_1, and let A be a subset of F_1. The functor $\mathrm{P}_2(x,y,A)$ yields an element of *Boolean* and is defined as follows:

(Def. 2) $$\mathrm{P}_2(x,y,A) = \begin{cases} \mathit{true}, & \text{if } y \in U(x) \text{ and } y \in A^{\mathrm{c}}, \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

We now state three propositions:

(6) For all elements x, y of F_1 and for every subset A of F_1 holds $\mathrm{P}_1(x,y,A) = \mathit{true}$ iff $y \in U(x)$ and $y \in A$.

(7) For all elements x, y of F_1 and for every subset A of F_1 holds $\mathrm{P}_2(x,y,A) = \mathit{true}$ iff $y \in U(x)$ and $y \in A^{\mathrm{c}}$.

(8) Let x be an element of F_1 and A be a subset of F_1. Then $x \in A^{\delta}$ if and only if there exist elements y_1, y_2 of F_1 such that $\mathrm{P}_1(x,y_1,A) = \mathit{true}$ and $\mathrm{P}_2(x,y_2,A) = \mathit{true}$.

Let F_1 be a non empty finite topology space, let x be an element of F_1, and let y be an element of F_1. The functor $\mathrm{P}_0(x,y)$ yields an element of *Boolean* and is defined as follows:

(Def. 3) $$\mathrm{P}_0(x,y) = \begin{cases} \mathit{true}, & \text{if } y \in U(x), \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

One can prove the following three propositions:

(9) For all elements x, y of F_1 holds $\mathrm{P}_0(x,y) = \mathit{true}$ iff $y \in U(x)$.

(10) Let x be an element of F_1 and A be a subset of F_1. Then $x \in A^{i}$ if and only if for every element y of F_1 such that $\mathrm{P}_0(x,y) = \mathit{true}$ holds $\mathrm{P}_1(x,y,A) = \mathit{true}$.

(11) Let x be an element of F_1 and A be a subset of F_1. Then $x \in A^{b}$ if and only if there exists an element y_1 of F_1 such that $\mathrm{P}_1(x,y_1,A) = \mathit{true}$.

Let F_1 be a non empty finite topology space, let x be an element of F_1, and let A be a subset of F_1. The functor $\mathrm{P}_{\mathrm{A}}(x,A)$ yields an element of *Boolean* and is defined by:

(Def. 4) $$\mathrm{P}_{\mathrm{A}}(x,A) = \begin{cases} \mathit{true}, & \text{if } x \in A, \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

We now state three propositions:

(12) For every element x of F_1 and for every subset A of F_1 holds $\mathrm{P}_{\mathrm{A}}(x,A) = \mathit{true}$ iff $x \in A$.

(13) Let x be an element of F_1 and A be a subset of F_1. Then $x \in A^{\delta_i}$ if and only if the following conditions are satisfied:

(i) there exist elements y_1, y_2 of F_1 such that $\mathrm{P}_1(x,y_1,A) = \mathit{true}$ and $\mathrm{P}_2(x,y_2,A) = \mathit{true}$, and

(ii) $\mathrm{P}_{\mathrm{A}}(x,A) = \mathit{true}$.

(14) Let x be an element of F_1 and A be a subset of F_1. Then $x \in A^{\delta_o}$ if and only if the following conditions are satisfied:

(i) there exist elements y_1, y_2 of F_1 such that $\mathrm{P}_1(x,y_1,A) = \mathit{true}$ and $\mathrm{P}_2(x,y_2,A) = \mathit{true}$, and

(ii) $\mathrm{P}_{\mathrm{A}}(x,A) = \mathit{false}$.

Let F_1 be a non empty finite topology space, let x be an element of F_1, and let y be an element of F_1. The functor $\mathrm{P}_{\mathrm{e}}(x,y)$ yields an element of *Boolean* and is defined as follows:

(Def. 5) $$\mathrm{P}_{\mathrm{e}}(x,y) = \begin{cases} \mathit{true}, & \text{if } x = y, \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

Next we state four propositions:

(15) For all elements x, y of F_1 holds $\mathrm{P_e}(x,y) = \mathit{true}$ iff $x = y$.

(16) Let x be an element of F_1 and A be a subset of F_1. Then $x \in A^s$ if and only if the following conditions are satisfied:

(i) $\mathrm{P_A}(x,A) = \mathit{true}$, and

(ii) it is not true that there exists an element y of F_1 such that $\mathrm{P_1}(x,y,A) = \mathit{true}$ and $\mathrm{P_e}(x,y) = \mathit{false}$.

(17) Let x be an element of F_1 and A be a subset of F_1. Then $x \in A^n$ if and only if the following conditions are satisfied:

(i) $\mathrm{P_A}(x,A) = \mathit{true}$, and

(ii) there exists an element y of F_1 such that $\mathrm{P_1}(x,y,A) = \mathit{true}$ and $\mathrm{P_e}(x,y) = \mathit{false}$.

(18) Let x be an element of F_1 and A be a subset of F_1. Then $x \in A^f$ if and only if there exists an element y of F_1 such that $\mathrm{P_A}(y,A) = \mathit{true}$ and $\mathrm{P_0}(y,x) = \mathit{true}$.

2. Formal Topological Spaces

We introduce formal topological spaces which are extensions of 1-sorted structure and are systems

$\langle$ a carrier, a Neighbour-map $\rangle$,

where the carrier is a set and the Neighbour-map is a function from the carrier into $2^{2^{\text{the carrier}}}$.

One can verify that there exists a formal topological space which is non empty and strict.

In the sequel T is a non empty topological structure, F_2 is a non empty formal topological space, and x is an element of F_2.

Let us consider F_2 and let x be an element of F_2. The functor $U_F(x)$ yielding a subset of $2^{\text{the carrier of } F_2}$ is defined as follows:

(Def. 6) $U_F(x) = (\text{the Neighbour-map of } F_2)(x)$.

Let us consider T. The functor NeighSpT yielding a non empty strict formal topological space is defined by the conditions (Def. 7).

(Def. 7)(i) The carrier of NeighSpT = the carrier of T, and

(ii) for every point x of NeighSpT holds $U_F(x) = \{V; V$ ranges over subsets of T: $V \in$ the topology of $T \wedge x \in V\}$.

In the sequel A, B, W, V denote subsets of F_2.

Let I_1 be a non empty formal topological space. We say that I_1 is filled if and only if:

(Def. 8) For every element x of I_1 and for every subset D of I_1 such that $D \in U_F(x)$ holds $x \in D$.

Let us consider F_2 and let us consider A. The functor $A^{F\delta}$ yielding a subset of F_2 is defined by:

(Def. 9) $A^{F\delta} = \{x : \bigwedge_W (W \in U_F(x) \Rightarrow W \text{ meets } A \wedge W \text{ meets } A^{\mathrm{c}})\}$.

Next we state the proposition

(20)[1] $x \in A^{F\delta}$ iff for every W such that $W \in U_F(x)$ holds W meets A and W meets A^{c}.

Let us consider F_2, A. The functor A^{F_b} yielding a subset of F_2 is defined as follows:

(Def. 10) $A^{F_b} = \{x : \bigwedge_W (W \in U_F(x) \Rightarrow W \text{ meets } A)\}$.

Next we state the proposition

(21) $x \in A^{F_b}$ iff for every W such that $W \in U_F(x)$ holds W meets A.

Let us consider F_2, A. The functor A^{F_i} yields a subset of F_2 and is defined by:

[1] The proposition (19) has been removed.

(Def. 11) $A^{F_i} = \{x : \bigvee_V (V \in U_F(x) \wedge V \subseteq A)\}$.

Next we state the proposition

(22) $x \in A^{F_i}$ iff there exists V such that $V \in U_F(x)$ and $V \subseteq A$.

Let us consider F_2, A. The functor A^{F_s} yielding a subset of F_2 is defined by:

(Def. 12) $A^{F_s} = \{x : x \in A \wedge \bigvee_V (V \in U_F(x) \wedge V \setminus \{x\} \text{ misses } A)\}$.

We now state the proposition

(23) $x \in A^{F_s}$ iff $x \in A$ and there exists V such that $V \in U_F(x)$ and $V \setminus \{x\}$ misses A.

Let us consider F_2, A. The functor $A^{F_{on}}$ yields a subset of F_2 and is defined by:

(Def. 13) $A^{F_{on}} = A \setminus A^{F_s}$.

One can prove the following propositions:

(24) $x \in A^{F_{on}}$ iff $x \in A$ and for every V such that $V \in U_F(x)$ holds $V \setminus \{x\}$ meets A.

(25) For every non empty formal topological space F_2 and for all subsets A, B of F_2 such that $A \subseteq B$ holds $A^{F_b} \subseteq B^{F_b}$.

(26) For every non empty formal topological space F_2 and for all subsets A, B of F_2 such that $A \subseteq B$ holds $A^{F_i} \subseteq B^{F_i}$.

(27) $A^{F_b} \cup B^{F_b} \subseteq A \cup B^{F_b}$.

(28) $A \cap B^{F_b} \subseteq A^{F_b} \cap B^{F_b}$.

(29) $A^{F_i} \cup B^{F_i} \subseteq A \cup B^{F_i}$.

(30) $A \cap B^{F_i} \subseteq A^{F_i} \cap B^{F_i}$.

(31) Let F_2 be a non empty formal topological space. Then for every element x of F_2 and for all subsets V_1, V_2 of F_2 such that $V_1 \in U_F(x)$ and $V_2 \in U_F(x)$ there exists a subset W of F_2 such that $W \in U_F(x)$ and $W \subseteq V_1 \cap V_2$ if and only if for all subsets A, B of F_2 holds $A \cup B^{F_b} = A^{F_b} \cup B^{F_b}$.

(32) Let F_2 be a non empty formal topological space. Then for every element x of F_2 and for all subsets V_1, V_2 of F_2 such that $V_1 \in U_F(x)$ and $V_2 \in U_F(x)$ there exists a subset W of F_2 such that $W \in U_F(x)$ and $W \subseteq V_1 \cap V_2$ if and only if for all subsets A, B of F_2 holds $A^{F_i} \cap B^{F_i} = A \cap B^{F_i}$.

(33) Let F_2 be a non empty formal topological space and A, B be subsets of F_2. Suppose that for every element x of F_2 and for all subsets V_1, V_2 of F_2 such that $V_1 \in U_F(x)$ and $V_2 \in U_F(x)$ there exists a subset W of F_2 such that $W \in U_F(x)$ and $W \subseteq V_1 \cap V_2$. Then $A \cup B^{F\delta} \subseteq A^{F\delta} \cup B^{F\delta}$.

(34) Let F_2 be a non empty formal topological space. Suppose F_2 is filled. Suppose that for all subsets A, B of F_2 holds $A \cup B^{F\delta} = A^{F\delta} \cup B^{F\delta}$. Let x be an element of F_2 and V_1, V_2 be subsets of F_2. If $V_1 \in U_F(x)$ and $V_2 \in U_F(x)$, then there exists a subset W of F_2 such that $W \in U_F(x)$ and $W \subseteq V_1 \cap V_2$.

(35) For every element x of F_2 and for every subset A of F_2 holds $x \in A^{F_s}$ iff $x \in A$ and $x \notin A \setminus \{x\}^{F_b}$.

(36) For every non empty formal topological space F_2 holds F_2 is filled iff for every subset A of F_2 holds $A \subseteq A^{F_b}$.

(37) For every non empty formal topological space F_2 holds F_2 is filled iff for every subset A of F_2 holds $A^{F_i} \subseteq A$.

(38) $(A^{\mathrm{c}F_b})^{\mathrm{c}} = A^{F_i}$.

(39) $(A^{\mathrm{c}F_i})^{\mathrm{c}} = A^{F_b}$.

(40) $A^{F\delta} = A^{F_b} \cap A^{\mathrm{c}F_b}$.

(41) $A^{F\delta} = A^{F_b} \cap (A^{F_i})^{\mathrm{c}}$.

(42) $A^{F\delta} = A^{\mathrm{c}F\delta}$.

(43) $A^{F\delta} = A^{F_b} \setminus A^{F_i}$.

Let us consider F_2 and let us consider A. The functor $A^{F\delta_i}$ yields a subset of F_2 and is defined as follows:

(Def. 14) $A^{F\delta_i} = A \cap A^{F\delta}$.

The functor $A^{F\delta_o}$ yields a subset of F_2 and is defined by:

(Def. 15) $A^{F\delta_o} = A^{\mathrm{c}} \cap A^{F\delta}$.

One can prove the following proposition

(44) $A^{F\delta} = A^{F\delta_i} \cup A^{F\delta_o}$.

Let us consider F_2 and let G be a subset of F_2. We say that G is open if and only if:

(Def. 16) $G = G^{F_i}$.

We say that G is closed if and only if:

(Def. 17) $G = G^{F_b}$.

We now state four propositions:

(45) If A^{c} is open, then A is closed.

(46) If A^{c} is closed, then A is open.

(47) Let F_2 be a non empty formal topological space and A, B be subsets of F_2. Suppose F_2 is filled. If for every element x of F_2 holds $\{x\} \in U_F(x)$, then $A \cap B^{F_b} = A^{F_b} \cap B^{F_b}$.

(48) Let F_2 be a non empty formal topological space and A, B be subsets of F_2. Suppose F_2 is filled. If for every element x of F_2 holds $\{x\} \in U_F(x)$, then $A^{F_i} \cup B^{F_i} = A \cup B^{F_i}$.

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