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Finite Sequences and Tuples of Elements of a Non-empty Sets

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Summary. The first part of the article is a continuation of [4]. Next, we define the identity sequence of natural numbers and the constant sequences. The main part of this article is the definition of tuples. The element of a set of all sequences of the length n of D is called a tuple of a non-empty set D and it is denoted by element of D^n. Also some basic facts about tuples of a non-empty set are proved.

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The articles [13], [12], [9], [16], [2], [3], [14], [11], [1], [15], [17], [6], [8], [7], [4], [5], and [10] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: i, j, l are natural numbers, k is a natural number, A, a, b, x, x_1, x_2, x_3 are sets, D, D', E are non empty sets, d, d_1, d_2, d_3 are elements of D, d', d'_1, d'_2, d'_3 are elements of D', and p, q, r are finite sequences.

One can prove the following propositions:

(1) $\min(i, j)$ is a natural number and $\max(i, j)$ is a natural number.

(2) If $l = \min(i, j)$, then $\operatorname{Seg} i \cap \operatorname{Seg} j = \operatorname{Seg} l$.

(3) If $i \leq j$, then $\max(0, i - j) = 0$.

(4) If $j \leq i$, then $\max(0, i - j) = i - j$.

(5) $\max(0, i - j)$ is a natural number.

(6) $\min(0, i) = 0$ and $\min(i, 0) = 0$ and $\max(0, i) = i$ and $\max(i, 0) = i$.

(8)[1] If $i \in \operatorname{Seg}(l + 1)$, then $i \in \operatorname{Seg} l$ or $i = l + 1$.

(9) If $i \in \operatorname{Seg} l$, then $i \in \operatorname{Seg}(l + j)$.

(10) If $\operatorname{len} p = i$ and $\operatorname{len} q = i$ and for every j such that $j \in \operatorname{Seg} i$ holds $p(j) = q(j)$, then $p = q$.

(11) If $b \in \operatorname{rng} p$, then there exists i such that $i \in \operatorname{dom} p$ and $p(i) = b$.

(13)[2] For every set D and for every finite sequence p of elements of D such that $i \in \operatorname{dom} p$ holds $p(i) \in D$.

[1] The proposition (7) has been removed.
[2] The proposition (12) has been removed.

(14) For every set D such that for every i such that $i \in \mathrm{dom}\, p$ holds $p(i) \in D$ holds p is a finite sequence of elements of D.

(15) $\langle d_1, d_2 \rangle$ is a finite sequence of elements of D.

(16) $\langle d_1, d_2, d_3 \rangle$ is a finite sequence of elements of D.

(18)[3] If $i \in \mathrm{dom}\, p$, then $i \in \mathrm{dom}(p \frown q)$.

(19) $\mathrm{len}(p \frown \langle a \rangle) = \mathrm{len}\, p + 1$.

(20) If $p \frown \langle a \rangle = q \frown \langle b \rangle$, then $p = q$ and $a = b$.

(21) If $\mathrm{len}\, p = i + 1$, then there exist q, a such that $p = q \frown \langle a \rangle$.

(22) Let p be a finite sequence of elements of D. Suppose $\mathrm{len}\, p \neq 0$. Then there exists a finite sequence q of elements of D and there exists d such that $p = q \frown \langle d \rangle$.

(23) If $q = p \restriction \mathrm{Seg}\, i$ and $\mathrm{len}\, p \leq i$, then $p = q$.

(24) If $q = p \restriction \mathrm{Seg}\, i$, then $\mathrm{len}\, q = \min(i, \mathrm{len}\, p)$.

(25) If $\mathrm{len}\, r = i + j$, then there exist p, q such that $\mathrm{len}\, p = i$ and $\mathrm{len}\, q = j$ and $r = p \frown q$.

(26) Let r be a finite sequence of elements of D. Suppose $\mathrm{len}\, r = i + j$. Then there exist finite sequences p, q of elements of D such that $\mathrm{len}\, p = i$ and $\mathrm{len}\, q = j$ and $r = p \frown q$.

In this article we present several logical schemes. The scheme *SeqLambdaD* deals with a natural number $\mathcal{A}$, a non empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

There exists a finite sequence z of elements of $\mathcal{B}$ such that $\mathrm{len}\, z = \mathcal{A}$ and for every j such that $j \in \mathrm{Seg}\, \mathcal{A}$ holds $z(j) = \mathcal{F}(j)$

for all values of the parameters.

The scheme *IndSeqD* deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every finite sequence p of elements of $\mathcal{A}$ holds $\mathcal{P}[p]$

provided the parameters meet the following conditions:

- $\mathcal{P}[\varepsilon_{\mathcal{A}}]$, and
- For every finite sequence p of elements of $\mathcal{A}$ and for every element x of $\mathcal{A}$ such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \frown \langle x \rangle]$.

One can prove the following propositions:

(27) For every non empty subset D' of D holds every finite sequence of elements of D' is a finite sequence of elements of D.

(28) Every function from $\mathrm{Seg}\, i$ into D is a finite sequence of elements of D.

(30)[4] Every finite sequence p of elements of D is a function from $\mathrm{dom}\, p$ into D.

(31) For every function f from $\mathbb{N}$ into D holds $f \restriction \mathrm{Seg}\, i$ is a finite sequence of elements of D.

(32) For every function f from $\mathbb{N}$ into D such that $q = f \restriction \mathrm{Seg}\, i$ holds $\mathrm{len}\, q = i$.

(33) For every function f such that $\mathrm{rng}\, p \subseteq \mathrm{dom}\, f$ and $q = f \cdot p$ holds $\mathrm{len}\, q = \mathrm{len}\, p$.

(34) Suppose $D = \mathrm{Seg}\, i$. Let p be a finite sequence and q be a finite sequence of elements of D. If $i \leq \mathrm{len}\, p$, then $p \cdot q$ is a finite sequence.

(35) Suppose $D = \mathrm{Seg}\, i$. Let p be a finite sequence of elements of D' and q be a finite sequence of elements of D. If $i \leq \mathrm{len}\, p$, then $p \cdot q$ is a finite sequence of elements of D'.

(36) Let A, D be sets, p be a finite sequence of elements of A, and f be a function from A into D. Then $f \cdot p$ is a finite sequence of elements of D.

[3] The proposition (17) has been removed.
[4] The proposition (29) has been removed.

(37) Let p be a finite sequence of elements of A and f be a function from A into D'. If $q = f \cdot p$, then $\operatorname{len} q = \operatorname{len} p$.

(38) For every function f from A into D' holds $f \cdot \varepsilon_A = \varepsilon_{D'}$.

(39) Let p be a finite sequence of elements of D and f be a function from D into D'. If $p = \langle x_1 \rangle$, then $f \cdot p = \langle f(x_1) \rangle$.

(40) Let p be a finite sequence of elements of D and f be a function from D into D'. If $p = \langle x_1, x_2 \rangle$, then $f \cdot p = \langle f(x_1), f(x_2) \rangle$.

(41) Let p be a finite sequence of elements of D and f be a function from D into D'. If $p = \langle x_1, x_2, x_3 \rangle$, then $f \cdot p = \langle f(x_1), f(x_2), f(x_3) \rangle$.

(42) For every function f from $\operatorname{Seg} i$ into $\operatorname{Seg} j$ such that if $j = 0$, then $i = 0$ and $j \leq \operatorname{len} p$ holds $p \cdot f$ is a finite sequence.

(43) For every function f from $\operatorname{Seg} i$ into $\operatorname{Seg} i$ such that $i \leq \operatorname{len} p$ holds $p \cdot f$ is a finite sequence.

(44) For every function f from $\operatorname{dom} p$ into $\operatorname{dom} p$ holds $p \cdot f$ is a finite sequence.

(45) For every function f from $\operatorname{Seg} i$ into $\operatorname{Seg} i$ such that $\operatorname{rng} f = \operatorname{Seg} i$ and $i \leq \operatorname{len} p$ and $q = p \cdot f$ holds $\operatorname{len} q = i$.

(46) For every function f from $\operatorname{dom} p$ into $\operatorname{dom} p$ such that $\operatorname{rng} f = \operatorname{dom} p$ and $q = p \cdot f$ holds $\operatorname{len} q = \operatorname{len} p$.

(47) For every permutation f of $\operatorname{Seg} i$ such that $i \leq \operatorname{len} p$ and $q = p \cdot f$ holds $\operatorname{len} q = i$.

(48) For every permutation f of $\operatorname{dom} p$ such that $q = p \cdot f$ holds $\operatorname{len} q = \operatorname{len} p$.

(49) Let p be a finite sequence of elements of D and f be a function from $\operatorname{Seg} i$ into $\operatorname{Seg} j$. Suppose if $j = 0$, then $i = 0$ and $j \leq \operatorname{len} p$. Then $p \cdot f$ is a finite sequence of elements of D.

(50) Let p be a finite sequence of elements of D and f be a function from $\operatorname{Seg} i$ into $\operatorname{Seg} i$. If $i \leq \operatorname{len} p$, then $p \cdot f$ is a finite sequence of elements of D.

(51) Let p be a finite sequence of elements of D and f be a function from $\operatorname{dom} p$ into $\operatorname{dom} p$. Then $p \cdot f$ is a finite sequence of elements of D.

(52) $\operatorname{id}_{\operatorname{Seg} k}$ is a finite sequence of elements of $\mathbb{N}$.

Let i be a natural number. The functor $\operatorname{idseq}(i)$ yielding a finite sequence is defined by:

(Def. 1) $\operatorname{idseq}(i) = \operatorname{id}_{\operatorname{Seg} i}$.

The following propositions are true:

(54)[5] $\operatorname{dom} \operatorname{idseq}(k) = \operatorname{Seg} k$.

(55) $\operatorname{len} \operatorname{idseq}(k) = k$.

(56) If $j \in \operatorname{Seg} i$, then $(\operatorname{idseq}(i))(j) = j$.

(57) If $i \neq 0$, then for every element k of $\operatorname{Seg} i$ holds $(\operatorname{idseq}(i))(k) = k$.

(58) $\operatorname{idseq}(0) = \emptyset$.

(59) $\operatorname{idseq}(1) = \langle 1 \rangle$.

(60) $\operatorname{idseq}(i+1) = (\operatorname{idseq}(i)) \frown \langle i+1 \rangle$.

(61) $\operatorname{idseq}(2) = \langle 1, 2 \rangle$.

[5] The proposition (53) has been removed.

(62) $\operatorname{idseq}(3) = \langle 1,2,3 \rangle$.

(63) $p \cdot \operatorname{idseq}(k) = p \restriction \operatorname{Seg} k$.

(64) If $\operatorname{len} p \leq k$, then $p \cdot \operatorname{idseq}(k) = p$.

(65) $\operatorname{idseq}(k)$ is a permutation of $\operatorname{Seg} k$.

(66) $\operatorname{Seg} k \longmapsto a$ is a finite sequence.

Let i be a natural number and let a be a set. The functor $i \mapsto a$ yielding a finite sequence is defined as follows:

(Def. 2) $i \mapsto a = \operatorname{Seg} i \longmapsto a$.

We now state a number of propositions:

(68)[6] $\operatorname{dom}(k \mapsto a) = \operatorname{Seg} k$.

(69) $\operatorname{len}(k \mapsto a) = k$.

(70) If $b \in \operatorname{Seg} k$, then $(k \mapsto a)(b) = a$.

(71) If $k \neq 0$, then for every element w of $\operatorname{Seg} k$ holds $(k \mapsto d)(w) = d$.

(72) $0 \mapsto a = \emptyset$.

(73) $1 \mapsto a = \langle a \rangle$.

(74) $(i+1) \mapsto a = (i \mapsto a) \frown \langle a \rangle$.

(75) $2 \mapsto a = \langle a,a \rangle$.

(76) $3 \mapsto a = \langle a,a,a \rangle$.

(77) $k \mapsto d$ is a finite sequence of elements of D.

(78) For every function F such that $[: \operatorname{rng} p, \operatorname{rng} q :] \subseteq \operatorname{dom} F$ holds $F^\circ(p, q)$ is a finite sequence.

(79) For every function F such that $[: \operatorname{rng} p, \operatorname{rng} q :] \subseteq \operatorname{dom} F$ and $r = F^\circ(p, q)$ holds $\operatorname{len} r = \min(\operatorname{len} p, \operatorname{len} q)$.

(80) For every function F such that $[: \{a\}, \operatorname{rng} p :] \subseteq \operatorname{dom} F$ holds $F^\circ(a, p)$ is a finite sequence.

(81) For every function F such that $[: \{a\}, \operatorname{rng} p :] \subseteq \operatorname{dom} F$ and $r = F^\circ(a, p)$ holds $\operatorname{len} r = \operatorname{len} p$.

(82) For every function F such that $[: \operatorname{rng} p, \{a\} :] \subseteq \operatorname{dom} F$ holds $F^\circ(p, a)$ is a finite sequence.

(83) For every function F such that $[: \operatorname{rng} p, \{a\} :] \subseteq \operatorname{dom} F$ and $r = F^\circ(p, a)$ holds $\operatorname{len} r = \operatorname{len} p$.

(84) Let F be a function from $[: D, D' :]$ into E, p be a finite sequence of elements of D, and q be a finite sequence of elements of D'. Then $F^\circ(p, q)$ is a finite sequence of elements of E.

(85) Let F be a function from $[: D, D' :]$ into E, p be a finite sequence of elements of D, and q be a finite sequence of elements of D'. If $r = F^\circ(p, q)$, then $\operatorname{len} r = \min(\operatorname{len} p, \operatorname{len} q)$.

(86) Let F be a function from $[: D, D' :]$ into E, p be a finite sequence of elements of D, and q be a finite sequence of elements of D'. If $\operatorname{len} p = \operatorname{len} q$ and $r = F^\circ(p, q)$, then $\operatorname{len} r = \operatorname{len} p$ and $\operatorname{len} r = \operatorname{len} q$.

(87) Let F be a function from $[: D, D' :]$ into E, p be a finite sequence of elements of D, and p' be a finite sequence of elements of D'. Then $F^\circ(\varepsilon_D, p') = \varepsilon_E$ and $F^\circ(p, \varepsilon_{D'}) = \varepsilon_E$.

[6] The proposition (67) has been removed.

(88) Let F be a function from $[: D, D' :]$ into E, p be a finite sequence of elements of D, and q be a finite sequence of elements of D'. If $p = \langle d_1 \rangle$ and $q = \langle d'_1 \rangle$, then $F^\circ(p, q) = \langle F(d_1, d'_1) \rangle$.

(89) Let F be a function from $[: D, D' :]$ into E, p be a finite sequence of elements of D, and q be a finite sequence of elements of D'. If $p = \langle d_1, d_2 \rangle$ and $q = \langle d'_1, d'_2 \rangle$, then $F^\circ(p, q) = \langle F(d_1, d'_1), F(d_2, d'_2) \rangle$.

(90) Let F be a function from $[: D, D' :]$ into E, p be a finite sequence of elements of D, and q be a finite sequence of elements of D'. If $p = \langle d_1, d_2, d_3 \rangle$ and $q = \langle d'_1, d'_2, d'_3 \rangle$, then $F^\circ(p, q) = \langle F(d_1, d'_1), F(d_2, d'_2), F(d_3, d'_3) \rangle$.

(91) Let F be a function from $[: D, D' :]$ into E and p be a finite sequence of elements of D'. Then $F^\circ(d, p)$ is a finite sequence of elements of E.

(92) Let F be a function from $[: D, D' :]$ into E and p be a finite sequence of elements of D'. If $r = F^\circ(d, p)$, then $\operatorname{len} r = \operatorname{len} p$.

(93) For every function F from $[: D, D' :]$ into E holds $F^\circ(d, \varepsilon_{D'}) = \varepsilon_E$.

(94) Let F be a function from $[: D, D' :]$ into E and p be a finite sequence of elements of D'. If $p = \langle d'_1 \rangle$, then $F^\circ(d, p) = \langle F(d, d'_1) \rangle$.

(95) Let F be a function from $[: D, D' :]$ into E and p be a finite sequence of elements of D'. If $p = \langle d'_1, d'_2 \rangle$, then $F^\circ(d, p) = \langle F(d, d'_1), F(d, d'_2) \rangle$.

(96) Let F be a function from $[: D, D' :]$ into E and p be a finite sequence of elements of D'. If $p = \langle d'_1, d'_2, d'_3 \rangle$, then $F^\circ(d, p) = \langle F(d, d'_1), F(d, d'_2), F(d, d'_3) \rangle$.

(97) Let F be a function from $[: D, D' :]$ into E and p be a finite sequence of elements of D. Then $F^\circ(p, d')$ is a finite sequence of elements of E.

(98) Let F be a function from $[: D, D' :]$ into E and p be a finite sequence of elements of D. If $r = F^\circ(p, d')$, then $\operatorname{len} r = \operatorname{len} p$.

(99) For every function F from $[: D, D' :]$ into E holds $F^\circ(\varepsilon_D, d') = \varepsilon_E$.

(100) Let F be a function from $[: D, D' :]$ into E and p be a finite sequence of elements of D. If $p = \langle d_1 \rangle$, then $F^\circ(p, d') = \langle F(d_1, d') \rangle$.

(101) Let F be a function from $[: D, D' :]$ into E and p be a finite sequence of elements of D. If $p = \langle d_1, d_2 \rangle$, then $F^\circ(p, d') = \langle F(d_1, d'), F(d_2, d') \rangle$.

(102) Let F be a function from $[: D, D' :]$ into E and p be a finite sequence of elements of D. If $p = \langle d_1, d_2, d_3 \rangle$, then $F^\circ(p, d') = \langle F(d_1, d'), F(d_2, d'), F(d_3, d') \rangle$.

Let D be a set. A set is called a set of finite sequences of D if:

(Def. 3) If $a \in$ it, then a is a finite sequence of elements of D.

Let D be a set. Note that there exists a set of finite sequences of D which is non empty.

Let D be a set. A non empty set of finite sequences of D is a non empty set of finite sequences of D.

We now state the proposition

(104)[7] For every set D holds D^* is a non empty set of finite sequences of D.

Let D be a set. Then D^* is a non empty set of finite sequences of D.

We now state the proposition

(105) For every set D and for every non empty set D' of finite sequences of D holds $D' \subseteq D^*$.

[7] The proposition (103) has been removed.

Let D be a set and let S be a non empty set of finite sequences of D. We see that the element of S is a finite sequence of elements of D.

The following proposition is true

(107)[8] For every non empty subset D' of D holds every non empty set of finite sequences of D' is a non empty set of finite sequences of D.

Let i be a natural number and let D be a set. The functor D^i yields a set of finite sequences of D and is defined by:

(Def. 4) $D^i = \{s; s$ ranges over elements of D^*: $\operatorname{len} s = i\}$.

Let i be a natural number and let us consider D. Note that D^i is non empty.

We now state a number of propositions:

(109)[9] For every element z of D^i holds $\operatorname{len} z = i$.

(110) For every set D holds every finite sequence z of elements of D is an element of $D^{\operatorname{len} z}$.

(111) $D^i = D^{\operatorname{Seg} i}$.

(112) For every set D holds $D^0 = \{\varepsilon_D\}$.

(113) For every set D and for every element z of D^0 holds $z = \varepsilon_D$.

(114) For every set D holds ε_D is an element of D^0.

(115) For every element z of D^0 and for every element t of D^i holds $z \frown t = t$ and $t \frown z = t$.

(116) $D^1 = \{\langle d \rangle\}$.

(117) For every element z of D^1 there exists d such that $z = \langle d \rangle$.

(118) $\langle d \rangle$ is an element of D^1.

(119) $D^2 = \{\langle d_1, d_2 \rangle\}$.

(120) For every element z of D^2 there exist d_1, d_2 such that $z = \langle d_1, d_2 \rangle$.

(121) $\langle d_1, d_2 \rangle$ is an element of D^2.

(122) $D^3 = \{\langle d_1, d_2, d_3 \rangle\}$.

(123) For every element z of D^3 there exist d_1, d_2, d_3 such that $z = \langle d_1, d_2, d_3 \rangle$.

(124) $\langle d_1, d_2, d_3 \rangle$ is an element of D^3.

(125) $D^{i+j} = \{z \frown t : z$ ranges over elements of D^i, t ranges over elements of $D^j\}$.

(126) For every element s of D^{i+j} there exists an element z of D^i and there exists an element t of D^j such that $s = z \frown t$.

(127) For every element z of D^i and for every element t of D^j holds $z \frown t$ is an element of D^{i+j}.

(128) $D^* = \bigcup\{D^i\}$.

(129) For every non empty subset D' of D holds every element of D'^i is an element of D^i.

(130) If $D^i = D^j$, then $i = j$.

(131) $\operatorname{idseq}(i)$ is an element of $\mathbb{N}^i$.

(132) $i \mapsto d$ is an element of D^i.

[8] The proposition (106) has been removed.
[9] The proposition (108) has been removed.

(133) For every element z of D^i and for every function f from D into D' holds $f \cdot z$ is an element of D'^i.

(134) Let z be an element of D^i and f be a function from $\mathrm{Seg}\, i$ into $\mathrm{Seg}\, i$. If $\mathrm{rng}\, f = \mathrm{Seg}\, i$, then $z \cdot f$ is an element of D^i.

(135) For every element z of D^i and for every permutation f of $\mathrm{Seg}\, i$ holds $z \cdot f$ is an element of D^i.

(136) For every element z of D^i and for every d holds $(z \frown \langle d \rangle)(i+1) = d$.

(137) For every element z of D^{i+1} there exists an element t of D^i and there exists d such that $z = t \frown \langle d \rangle$.

(138) For every element z of D^i holds $z \cdot \mathrm{idseq}(i) = z$.

(139) For all elements z_1, z_2 of D^i such that for every j such that $j \in \mathrm{Seg}\, i$ holds $z_1(j) = z_2(j)$ holds $z_1 = z_2$.

(140) Let F be a function from $[:D, D':]$ into E, z_1 be an element of D^i, and z_2 be an element of D'^i. Then $F^\circ(z_1, z_2)$ is an element of E^i.

(141) For every function F from $[:D, D':]$ into E and for every element z of D'^i holds $F^\circ(d, z)$ is an element of E^i.

(142) For every function F from $[:D, D':]$ into E and for every element z of D^i holds $F^\circ(z, d')$ is an element of E^i.

(143) $(i + j) \mapsto x = (i \mapsto x) \frown (j \mapsto x)$.

(144) For all i, D and for every element x of D^i holds $\mathrm{dom}\, x = \mathrm{Seg}\, i$.

(145) For every function f and for all sets x, y such that $x \in \mathrm{dom}\, f$ and $y \in \mathrm{dom}\, f$ holds $f \cdot \langle x, y \rangle = \langle f(x), f(y) \rangle$.

(146) For every function f and for all sets x, y, z such that $x \in \mathrm{dom}\, f$ and $y \in \mathrm{dom}\, f$ and $z \in \mathrm{dom}\, f$ holds $f \cdot \langle x, y, z \rangle = \langle f(x), f(y), f(z) \rangle$.

(147) $\mathrm{rng} \langle x_1, x_2 \rangle = \{x_1, x_2\}$.

(148) $\mathrm{rng} \langle x_1, x_2, x_3 \rangle = \{x_1, x_2, x_3\}$.

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