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Real Function Differentiability — Part II

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Summary. A continuation of [16]. We prove equivalent definition of the derivative of the real function at the point and theorems about derivative of composite functions, inverse function and derivative of quotient of two functions. At the beginning of the paper a few facts which rather belong to [8], [9] and [7] are proved.

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The articles [18], [21], [2], [19], [6], [1], [3], [4], [10], [22], [8], [15], [5], [20], [12], [13], [17], [14], [16], [11], and [9] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: x_0, r, r_1, r_2, g, p are real numbers, n, m are natural numbers, a, b, d are sequences of real numbers, h, h_1, h_2 are convergent to 0 sequences of real numbers, c is a constant sequence of real numbers, A is an open subset of $\mathbb{R}$, and f, f_1, f_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Let us consider h. Observe that $-h$ is convergent to 0.

One can prove the following propositions:

(1) Suppose a is convergent and b is convergent and $\lim a = \lim b$ and for every n holds $d(2 \cdot n) = a(n)$ and $d(2 \cdot n+1) = b(n)$. Then d is convergent and $\lim d = \lim a$.

(2) If for every n holds $a(n) = 2 \cdot n$, then a is increasing and natural-yielding.

(3) If for every n holds $a(n) = 2 \cdot n+1$, then a is increasing and natural-yielding.

(4) If $\operatorname{rng} c = \{x_0\}$, then c is convergent and $\lim c = x_0$ and $h+c$ is convergent and $\lim(h+c) = x_0$.

(5) If $\operatorname{rng} a = \{r\}$ and $\operatorname{rng} b = \{r\}$, then $a = b$.

(6) If a is a subsequence of h, then a is a convergent to 0 sequence of real numbers.

(7) Suppose that for all h, c such that $\operatorname{rng} c = \{g\}$ and $\operatorname{rng}(h+c) \subseteq \operatorname{dom} f$ and $\{g\} \subseteq \operatorname{dom} f$ holds $h^{-1}\,(f \cdot (h+c) - f \cdot c)$ is convergent. Let given h_1, h_2, c. Suppose $\operatorname{rng} c = \{g\}$ and $\operatorname{rng}(h_1+c) \subseteq \operatorname{dom} f$ and $\operatorname{rng}(h_2+c) \subseteq \operatorname{dom} f$ and $\{g\} \subseteq \operatorname{dom} f$. Then $\lim({h_1}^{-1}\,(f \cdot (h_1+c) - f \cdot c)) = \lim({h_2}^{-1}\,(f \cdot (h_2+c) - f \cdot c))$.

(8) If there exists a neighbourhood N of r such that $N \subseteq \operatorname{dom} f$, then there exist h, c such that $\operatorname{rng} c = \{r\}$ and $\operatorname{rng}(h+c) \subseteq \operatorname{dom} f$ and $\{r\} \subseteq \operatorname{dom} f$.

(9) If $\operatorname{rng} a \subseteq \operatorname{dom}(f_2 \cdot f_1)$, then $\operatorname{rng} a \subseteq \operatorname{dom} f_1$ and $\operatorname{rng}(f_1 \cdot a) \subseteq \operatorname{dom} f_2$.

The scheme *ExInc Seq of Nat* deals with a sequence $\mathcal{A}$ of real numbers and a unary predicate $\mathcal{P}$, and states that:

There exists an increasing sequence q of naturals such that for every n holds $\mathcal{P}[(\mathcal{A} \cdot q)(n)]$ and for every n such that for every r such that $r = \mathcal{A}(n)$ holds $\mathcal{P}[r]$ there exists m such that $n = q(m)$

provided the parameters meet the following condition:

- For every n there exists m such that $n \leq m$ and $\mathcal{P}[\mathcal{A}(m)]$.

We now state a number of propositions:

(10) Suppose $f(x_0) \neq r$ and f is differentiable in x_0. Then there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and for every g such that $g \in N$ holds $f(g) \neq r$.

(11) f is differentiable in x_0 if and only if the following conditions are satisfied:

(i) there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$, and

(ii) for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h+c) \subseteq \operatorname{dom} f$ holds $h^{-1}\,(f \cdot (h+c) - f \cdot c)$ is convergent.

(12) f is differentiable in x_0 and $f'(x_0) = g$ if and only if the following conditions are satisfied:

(i) there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$, and

(ii) for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h+c) \subseteq \operatorname{dom} f$ holds $h^{-1}\,(f \cdot (h+c) - f \cdot c)$ is convergent and $\lim(h^{-1}\,(f \cdot (h+c) - f \cdot c)) = g$.

(13) If f_1 is differentiable in x_0 and f_2 is differentiable in $f_1(x_0)$, then $f_2 \cdot f_1$ is differentiable in x_0 and $(f_2 \cdot f_1)'(x_0) = {f_2}'(f_1(x_0)) \cdot {f_1}'(x_0)$.

(14) Suppose $f_2(x_0) \neq 0$ and f_1 is differentiable in x_0 and f_2 is differentiable in x_0. Then $\frac{f_1}{f_2}$ is differentiable in x_0 and $(\frac{f_1}{f_2})'(x_0) = \frac{{f_1}'(x_0) \cdot f_2(x_0) - {f_2}'(x_0) \cdot f_1(x_0)}{f_2(x_0)^2}$.

(15) If $f(x_0) \neq 0$ and f is differentiable in x_0, then $\frac{1}{f}$ is differentiable in x_0 and $(\frac{1}{f})'(x_0) = -\frac{f'(x_0)}{f(x_0)^2}$.

(16) If f is differentiable on A, then $f{\restriction}A$ is differentiable on A and $f'_{\restriction A} = (f{\restriction}A)'_{\restriction A}$.

(17) If f_1 is differentiable on A and f_2 is differentiable on A, then $f_1 + f_2$ is differentiable on A and $(f_1 + f_2)'_{\restriction A} = (f_1)'_{\restriction A} + (f_2)'_{\restriction A}$.

(18) If f_1 is differentiable on A and f_2 is differentiable on A, then $f_1 - f_2$ is differentiable on A and $(f_1 - f_2)'_{\restriction A} = (f_1)'_{\restriction A} - (f_2)'_{\restriction A}$.

(19) If f is differentiable on A, then $r\,f$ is differentiable on A and $(r\,f)'_{\restriction A} = r\,f'_{\restriction A}$.

(20) If f_1 is differentiable on A and f_2 is differentiable on A, then $f_1\,f_2$ is differentiable on A and $(f_1\,f_2)'_{\restriction A} = (f_1)'_{\restriction A}\,f_2 + f_1\,(f_2)'_{\restriction A}$.

(21) Suppose f_1 is differentiable on A and f_2 is differentiable on A and for every x_0 such that $x_0 \in A$ holds $f_2(x_0) \neq 0$. Then $\frac{f_1}{f_2}$ is differentiable on A and $(\frac{f_1}{f_2})'_{\restriction A} = \frac{(f_1)'_{\restriction A}\,f_2 - (f_2)'_{\restriction A}\,f_1}{f_2\,f_2}$.

(22) Suppose f is differentiable on A and for every x_0 such that $x_0 \in A$ holds $f(x_0) \neq 0$. Then $\frac{1}{f}$ is differentiable on A and $(\frac{1}{f})'_{\restriction A} = -\frac{f'_{\restriction A}}{f\,f}$.

(23) Suppose f_1 is differentiable on A and ${f_1}^{\circ}A$ is an open subset of $\mathbb{R}$ and f_2 is differentiable on ${f_1}^{\circ}A$. Then $f_2 \cdot f_1$ is differentiable on A and $(f_2 \cdot f_1)'_{\restriction A} = ((f_2)'_{\restriction {f_1}^{\circ}A} \cdot f_1)\,(f_1)'_{\restriction A}$.

(24) Suppose $A \subseteq \operatorname{dom} f$ and for all r, p such that $r \in A$ and $p \in A$ holds $|f(r) - f(p)| \leq (r-p)^2$. Then f is differentiable on A and for every x_0 such that $x_0 \in A$ holds $f'(x_0) = 0$.

(25) Suppose for all r_1, r_2 such that $r_1 \in]p,g[$ and $r_2 \in]p,g[$ holds $|f(r_1) - f(r_2)| \leq (r_1 - r_2)^2$ and $p < g$ and $]p,g[\subseteq \mathrm{dom} f$. Then f is differentiable on $]p,g[$ and a constant on $]p,g[$.

(26) Suppose $]-\infty,r[\subseteq \mathrm{dom} f$ and for all r_1, r_2 such that $r_1 \in]-\infty,r[$ and $r_2 \in]-\infty,r[$ holds $|f(r_1) - f(r_2)| \leq (r_1 - r_2)^2$. Then f is differentiable on $]-\infty,r[$ and a constant on $]-\infty,r[$.

(27) Suppose $]r,+\infty[\subseteq \mathrm{dom} f$ and for all r_1, r_2 such that $r_1 \in]r,+\infty[$ and $r_2 \in]r,+\infty[$ holds $|f(r_1) - f(r_2)| \leq (r_1 - r_2)^2$. Then f is differentiable on $]r,+\infty[$ and a constant on $]r,+\infty[$.

(28) If f is total and for all r_1, r_2 holds $|f(r_1) - f(r_2)| \leq (r_1 - r_2)^2$, then f is differentiable on $\Omega_{\mathbb{R}}$ and a constant on $\Omega_{\mathbb{R}}$.

(29) Suppose f is differentiable on $]-\infty,r[$ and for every x_0 such that $x_0 \in]-\infty,r[$ holds $0 < f'(x_0)$. Then f is increasing on $]-\infty,r[$ and $f{\restriction}]-\infty,r[$ is one-to-one.

(30) Suppose f is differentiable on $]-\infty,r[$ and for every x_0 such that $x_0 \in]-\infty,r[$ holds $f'(x_0) < 0$. Then f is decreasing on $]-\infty,r[$ and $f{\restriction}]-\infty,r[$ is one-to-one.

(31) If f is differentiable on $]-\infty,r[$ and for every x_0 such that $x_0 \in]-\infty,r[$ holds $0 \leq f'(x_0)$, then f is non-decreasing on $]-\infty,r[$.

(32) If f is differentiable on $]-\infty,r[$ and for every x_0 such that $x_0 \in]-\infty,r[$ holds $f'(x_0) \leq 0$, then f is non increasing on $]-\infty,r[$.

(33) Suppose f is differentiable on $]r,+\infty[$ and for every x_0 such that $x_0 \in]r,+\infty[$ holds $0 < f'(x_0)$. Then f is increasing on $]r,+\infty[$ and $f{\restriction}]r,+\infty[$ is one-to-one.

(34) Suppose f is differentiable on $]r,+\infty[$ and for every x_0 such that $x_0 \in]r,+\infty[$ holds $f'(x_0) < 0$. Then f is decreasing on $]r,+\infty[$ and $f{\restriction}]r,+\infty[$ is one-to-one.

(35) If f is differentiable on $]r,+\infty[$ and for every x_0 such that $x_0 \in]r,+\infty[$ holds $0 \leq f'(x_0)$, then f is non-decreasing on $]r,+\infty[$.

(36) If f is differentiable on $]r,+\infty[$ and for every x_0 such that $x_0 \in]r,+\infty[$ holds $f'(x_0) \leq 0$, then f is non increasing on $]r,+\infty[$.

(37) If f is differentiable on $\Omega_{\mathbb{R}}$ and for every x_0 holds $0 < f'(x_0)$, then f is increasing on $\Omega_{\mathbb{R}}$ and one-to-one.

(38) If f is differentiable on $\Omega_{\mathbb{R}}$ and for every x_0 holds $f'(x_0) < 0$, then f is decreasing on $\Omega_{\mathbb{R}}$ and one-to-one.

(39) If f is differentiable on $\Omega_{\mathbb{R}}$ and for every x_0 holds $0 \leq f'(x_0)$, then f is non-decreasing on $\Omega_{\mathbb{R}}$.

(40) If f is differentiable on $\Omega_{\mathbb{R}}$ and for every x_0 holds $f'(x_0) \leq 0$, then f is non increasing on $\Omega_{\mathbb{R}}$.

(41) Suppose f is differentiable on $]p,g[$ but for every x_0 such that $x_0 \in]p,g[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in]p,g[$ holds $f'(x_0) < 0$. Then $\mathrm{rng}(f{\restriction}]p,g[)$ is open.

(42) Suppose f is differentiable on $]-\infty,p[$ but for every x_0 such that $x_0 \in]-\infty,p[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in]-\infty,p[$ holds $f'(x_0) < 0$. Then $\mathrm{rng}(f{\restriction}]-\infty,p[)$ is open.

(43) Suppose f is differentiable on $]p,+\infty[$ but for every x_0 such that $x_0 \in]p,+\infty[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in]p,+\infty[$ holds $f'(x_0) < 0$. Then $\mathrm{rng}(f{\restriction}]p,+\infty[)$ is open.

(44) If f is differentiable on $\Omega_{\mathbb{R}}$ and if for every x_0 holds $0 < f'(x_0)$ or for every x_0 holds $f'(x_0) < 0$, then $\mathrm{rng} f$ is open.

(45) Let f be an one-to-one partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is differentiable on $\Omega_{\mathbb{R}}$ but for every x_0 holds $0 < f'(x_0)$ or for every x_0 holds $f'(x_0) < 0$. Then f is one-to-one and f^{-1} is differentiable on $\mathrm{dom}(f^{-1})$ and for every x_0 such that $x_0 \in \mathrm{dom}(f^{-1})$ holds $(f^{-1})'(x_0) = \frac{1}{f'(f^{-1}(x_0))}$.

(46) Let f be an one-to-one partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is differentiable on $]-\infty,p[$ but for every x_0 such that $x_0 \in]-\infty,p[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in]-\infty,p[$ holds $f'(x_0) < 0$. Then $f\restriction]-\infty,p[$ is one-to-one and $(f\restriction]-\infty,p[)^{-1}$ is differentiable on $\mathrm{dom}((f\restriction]-\infty,p[)^{-1})$ and for every x_0 such that $x_0 \in \mathrm{dom}((f\restriction]-\infty,p[)^{-1})$ holds $((f\restriction]-\infty,p[)^{-1})'(x_0) = \frac{1}{f'((f\restriction]-\infty,p[)^{-1}(x_0))}$.

(47) Let f be an one-to-one partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is differentiable on $]p,+\infty[$ but for every x_0 such that $x_0 \in]p,+\infty[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in]p,+\infty[$ holds $f'(x_0) < 0$. Then $f\restriction]p,+\infty[$ is one-to-one and $(f\restriction]p,+\infty[)^{-1}$ is differentiable on $\mathrm{dom}((f\restriction]p,+\infty[)^{-1})$ and for every x_0 such that $x_0 \in \mathrm{dom}((f\restriction]p,+\infty[)^{-1})$ holds $((f\restriction]p,+\infty[)^{-1})'(x_0) = \frac{1}{f'((f\restriction]p,+\infty[)^{-1}(x_0))}$.

(48) Let f be an one-to-one partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is differentiable on $]p,g[$ but for every x_0 such that $x_0 \in]p,g[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in]p,g[$ holds $f'(x_0) < 0$. Then

(i) $f\restriction]p,g[$ is one-to-one,

(ii) $(f\restriction]p,g[)^{-1}$ is differentiable on $\mathrm{dom}((f\restriction]p,g[)^{-1})$, and

(iii) for every x_0 such that $x_0 \in \mathrm{dom}((f\restriction]p,g[)^{-1})$ holds $((f\restriction]p,g[)^{-1})'(x_0) = \frac{1}{f'((f\restriction]p,g[)^{-1}(x_0))}$.

(49) Suppose f is differentiable in x_0. Let given h, c. Suppose $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h+c) \subseteq \mathrm{dom} f$ and $\mathrm{rng}(-h+c) \subseteq \mathrm{dom} f$. Then $(2\,h)^{-1}\,(f\cdot(c+h) - f\cdot(c-h))$ is convergent and $\lim((2\,h)^{-1}\,(f\cdot(c+h) - f\cdot(c-h))) = f'(x_0)$.

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