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Inst. of Computer Science, Univ. of Białystok

# Real Function Uniform Continuity[1]

Jarosław Kotowicz
Warsaw University
Białystok

Konrad Raczkowski
Warsaw University
Białystok

**Summary.** The uniform continuity for real functions is introduced. More theorems concerning continuous functions are given. (See [10]) The Darboux Theorem is exposed. Algebraic features for uniformly continuous functions are presented. Various facts, e.g., a continuous function on a compact set is uniformly continuous are proved.

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The articles [12], [14], [1], [13], [3], [2], [9], [15], [5], [4], [6], [7], [8], [11], and [10] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: $X$, $X_1$, $Z$, $Z_1$ denote sets, $s$, $g$, $r$, $p$, $x_1$, $x_2$ denote real numbers, $Y$ denotes a subset of $\mathbb{R}$, and $f$, $f_1$, $f_2$ denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Let us consider $f$, $X$. We say that $f$ is uniformly continuous on $X$ if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every $r$ such that $0 < r$ there exists $s$ such that $0 < s$ and for all $x_1$, $x_2$ such that $x_1 \in X$ and $x_2 \in X$ and $|x_1 - x_2| < s$ holds $|f(x_1) - f(x_2)| < r$.

One can prove the following propositions:

(2)[1] If $f$ is uniformly continuous on $X$ and $X_1 \subseteq X$, then $f$ is uniformly continuous on $X_1$.

(3) If $f_1$ is uniformly continuous on $X$ and $f_2$ is uniformly continuous on $X_1$, then $f_1 + f_2$ is uniformly continuous on $X \cap X_1$.

(4) If $f_1$ is uniformly continuous on $X$ and $f_2$ is uniformly continuous on $X_1$, then $f_1 - f_2$ is uniformly continuous on $X \cap X_1$.

(5) If $f$ is uniformly continuous on $X$, then $p\,f$ is uniformly continuous on $X$.

(6) If $f$ is uniformly continuous on $X$, then $-f$ is uniformly continuous on $X$.

(7) If $f$ is uniformly continuous on $X$, then $|f|$ is uniformly continuous on $X$.

(8) Suppose $f_1$ is uniformly continuous on $X$ and $f_2$ is uniformly continuous on $X_1$ and $f_1$ is bounded on $Z$ and $f_2$ is bounded on $Z_1$. Then $f_1\, f_2$ is uniformly continuous on $X \cap Z \cap X_1 \cap Z_1$.

(9) If $f$ is uniformly continuous on $X$, then $f$ is continuous on $X$.

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[1] The proposition (1) has been removed.

(10) If $f$ is Lipschitzian on $X$, then $f$ is uniformly continuous on $X$.

(11) For all $f$, $Y$ such that $Y$ is compact and $f$ is continuous on $Y$ holds $f$ is uniformly continuous on $Y$.

(13)[2] If $Y \subseteq \mathrm{dom} f$ and $Y$ is compact and $f$ is uniformly continuous on $Y$, then $f^{\circ}Y$ is compact.

(14) Let given $f$, $Y$. Suppose $Y \neq \emptyset$ and $Y \subseteq \mathrm{dom} f$ and $Y$ is compact and $f$ is uniformly continuous on $Y$. Then there exist $x_1$, $x_2$ such that $x_1 \in Y$ and $x_2 \in Y$ and $f(x_1) = \sup(f^{\circ}Y)$ and $f(x_2) = \inf(f^{\circ}Y)$.

(15) If $X \subseteq \mathrm{dom} f$ and $f$ is a constant on $X$, then $f$ is uniformly continuous on $X$.

(16) If $p \leq g$ and $f$ is continuous on $[p,g]$, then for every $r$ such that $r \in [f(p), f(g)] \cup [f(g), f(p)]$ there exists $s$ such that $s \in [p,g]$ and $r = f(s)$.

(17) If $p \leq g$ and $f$ is continuous on $[p,g]$, then for every $r$ such that $r \in [\inf(f^{\circ}[p,g]), \sup(f^{\circ}[p,g])]$ there exists $s$ such that $s \in [p,g]$ and $r = f(s)$.

(18) If $f$ is one-to-one and $p \leq g$ and $f$ is continuous on $[p,g]$, then $f$ is increasing on $[p,g]$ and decreasing on $[p,g]$.

(19) Suppose $f$ is one-to-one and $p \leq g$ and $f$ is continuous on $[p,g]$. Then $\inf(f^{\circ}[p,g]) = f(p)$ and $\sup(f^{\circ}[p,g]) = f(g)$ or $\inf(f^{\circ}[p,g]) = f(g)$ and $\sup(f^{\circ}[p,g]) = f(p)$.

(20) If $p \leq g$ and $f$ is continuous on $[p,g]$, then $f^{\circ}[p,g] = [\inf(f^{\circ}[p,g]), \sup(f^{\circ}[p,g])]$.

(21) Let $f$ be an one-to-one partial function from $\mathbb{R}$ to $\mathbb{R}$. If $p \leq g$ and $f$ is continuous on $[p,g]$, then $f^{-1}$ is continuous on $[\inf(f^{\circ}[p,g]), \sup(f^{\circ}[p,g])]$.

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[2] The proposition (12) has been removed.

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