JOURNAL OF FORMALIZED MATHEMATICS
Volume 7, Released 1995, Published 2003
Inst. of Computer Science, Univ. of Białystok

Full Adder Circuit. Part I[1]

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Summary. We continue the formalisation of circuits started by Piotr Rudnicki, Andrzej Trybulec, Pauline Kawamoto, and the second author in [12], [13], [11], [14]. The first step in proving properties of full n-bit adder circuit, i.e. 1-bit adder, is presented. We employ the notation of combining circuits introduced in [10].

MML Identifier: `FACIRC_1`.

WWW: `http://mizar.org/JFM/Vol7/facirc_1.html`

The articles [17], [16], [21], [20], [1], [18], [22], [4], [5], [3], [9], [7], [8], [23], [15], [2], [6], [19], [13], [14], and [10] provide the notation and terminology for this paper.

1. Combining of Many Sorted Signatures

Let I_1 be a set. We say that I_1 is pair if and only if:

(Def. 1) There exist sets x, y such that $I_1 = \langle x, y\rangle$.

Let us note that every set which is pair is also non empty.
Let x, y be sets. Note that $\langle x, y\rangle$ is pair.
One can verify that there exists a set which is pair and there exists a set which is non pair.
Let us observe that every natural number is non pair.
Let I_1 be a set. We say that I_1 has a pair if and only if:

(Def. 2) There exists a pair set x such that $x \in I_1$.

We introduce I_1 has no pairs as an antonym of I_1 has a pair.

Observe that every set which is empty has also no pairs. Let x be a non pair set. Observe that $\{x\}$ has no pairs. Let y be a non pair set. Note that $\{x, y\}$ has no pairs. Let z be a non pair set. One can verify that $\{x, y, z\}$ has no pairs.

Let us observe that there exists a non empty set which has no pairs.

Let X, Y be sets with no pairs. One can check that $X \cup Y$ has no pairs.

Let X be a set with no pairs and let Y be a set. One can check the following observations:

* $X \setminus Y$ has no pairs,
* $X \cap Y$ has no pairs, and
* $Y \cap X$ has no pairs.

[1]This work was written while the first author visited Shinshu University, July–August 1994.

Let x be a pair set. Observe that $\{x\}$ is relation-like. Let y be a pair set. One can verify that $\{x,y\}$ is relation-like. Let z be a pair set. Note that $\{x,y,z\}$ is relation-like.

One can verify that every set which is relation-like and has no pairs is also empty.

Let I_1 be a function. We say that I_1 is nonpair yielding if and only if:

(Def. 3) For every set x such that $x \in \mathrm{dom}\, I_1$ holds $I_1(x)$ is non pair.

Let x be a non pair set. Note that $\langle x \rangle$ is nonpair yielding. Let y be a non pair set. One can verify that $\langle x, y \rangle$ is nonpair yielding. Let z be a non pair set. Observe that $\langle x, y, z \rangle$ is nonpair yielding.

Next we state the proposition

(1) For every function f such that f is nonpair yielding holds $\mathrm{rng}\, f$ has no pairs.

Let n be a natural number. Observe that there exists a finite sequence with length n which is one-to-one and nonpair yielding.

Let us observe that there exists a finite sequence which is one-to-one and nonpair yielding.

Let f be a nonpair yielding function. Observe that $\mathrm{rng}\, f$ has no pairs.

One can prove the following propositions:

(2) Let S_1, S_2 be non empty many sorted signatures. Suppose $S_1 \approx S_2$ and $\mathrm{InnerVertices}(S_1)$ is a binary relation and $\mathrm{InnerVertices}(S_2)$ is a binary relation. Then $\mathrm{InnerVertices}(S_1 {+\cdot} S_2)$ is a binary relation.

(3) Let S_1, S_2 be unsplit non empty many sorted signatures with arity held in gates. Suppose $\mathrm{InnerVertices}(S_1)$ is a binary relation and $\mathrm{InnerVertices}(S_2)$ is a binary relation. Then $\mathrm{InnerVertices}(S_1 {+\cdot} S_2)$ is a binary relation.

(4) For all non empty many sorted signatures S_1, S_2 such that $S_1 \approx S_2$ and $\mathrm{InnerVertices}(S_2)$ misses $\mathrm{InputVertices}(S_1)$ holds $\mathrm{InputVertices}(S_1) \subseteq \mathrm{InputVertices}(S_1 {+\cdot} S_2)$ and $\mathrm{InputVertices}(S_1 {+\cdot} S_2) = \mathrm{InputVertices}(S_1) \cup (\mathrm{InputVertices}(S_2) \setminus \mathrm{InnerVertices}(S_1))$.

(5) For all sets X, R such that X has no pairs and R is a binary relation holds X misses R.

(6) Let S_1, S_2 be unsplit non empty many sorted signatures with arity held in gates. Suppose $\mathrm{InputVertices}(S_1)$ has no pairs and $\mathrm{InnerVertices}(S_2)$ is a binary relation. Then $\mathrm{InputVertices}(S_1) \subseteq \mathrm{InputVertices}(S_1 {+\cdot} S_2)$ and $\mathrm{InputVertices}(S_1 {+\cdot} S_2) = \mathrm{InputVertices}(S_1) \cup (\mathrm{InputVertices}(S_2) \setminus \mathrm{InnerVertices}(S_1))$.

(7) Let S_1, S_2 be unsplit non empty many sorted signatures with arity held in gates. Suppose $\mathrm{InputVertices}(S_1)$ has no pairs and $\mathrm{InnerVertices}(S_1)$ is a binary relation and $\mathrm{InputVertices}(S_2)$ has no pairs and $\mathrm{InnerVertices}(S_2)$ is a binary relation. Then $\mathrm{InputVertices}(S_1 {+\cdot} S_2) = \mathrm{InputVertices}(S_1) \cup \mathrm{InputVertices}(S_2)$.

(8) For all non empty many sorted signatures S_1, S_2 such that $S_1 \approx S_2$ and $\mathrm{InputVertices}(S_1)$ has no pairs and $\mathrm{InputVertices}(S_2)$ has no pairs holds $\mathrm{InputVertices}(S_1 {+\cdot} S_2)$ has no pairs.

(9) Let S_1, S_2 be unsplit non empty many sorted signatures with arity held in gates. If $\mathrm{InputVertices}(S_1)$ has no pairs and $\mathrm{InputVertices}(S_2)$ has no pairs, then $\mathrm{InputVertices}(S_1 {+\cdot} S_2)$ has no pairs.

2. Combining of Circuits

In this article we present several logical schemes. The scheme *2AryBooleEx* deals with a binary functor $\mathcal{F}$ yielding an element of *Boolean*, and states that:

There exists a function f from $\mathit{Boolean}^2$ into *Boolean* such that for all elements x, y of *Boolean* holds $f(\langle x, y \rangle) = \mathcal{F}(x, y)$

for all values of the parameter.

The scheme *2AryBooleUniq* deals with a binary functor $\mathcal{F}$ yielding an element of *Boolean*, and states that:

Let f_1, f_2 be functions from *Boolean*^{2} into *Boolean*. Suppose for all elements x, y of *Boolean* holds $f_1(\langle x,y\rangle) = \mathcal{F}(x,y)$ and for all elements x, y of *Boolean* holds $f_2(\langle x, y\rangle) = \mathcal{F}(x,y)$. Then $f_1 = f_2$

for all values of the parameter.

The scheme *2AryBooleDef* deals with a binary functor $\mathcal{F}$ yielding an element of *Boolean*, and states that:

(i) There exists a function f from *Boolean*^{2} into *Boolean* such that for all elements x, y of *Boolean* holds $f(\langle x,y\rangle) = \mathcal{F}(x,y)$, and

(ii) for all functions f_1, f_2 from *Boolean*^{2} into *Boolean* such that for all elements x, y of *Boolean* holds $f_1(\langle x,y\rangle) = \mathcal{F}(x,y)$ and for all elements x, y of *Boolean* holds $f_2(\langle x,y\rangle) = \mathcal{F}(x,y)$ holds $f_1 = f_2$

for all values of the parameter.

The scheme *3AryBooleEx* deals with a ternary functor $\mathcal{F}$ yielding an element of *Boolean*, and states that:

There exists a function f from *Boolean*^{3} into *Boolean* such that for all elements x, y, z of *Boolean* holds $f(\langle x,y,z\rangle) = \mathcal{F}(x,y,z)$

for all values of the parameter.

The scheme *3AryBooleUniq* deals with a ternary functor $\mathcal{F}$ yielding an element of *Boolean*, and states that:

Let f_1, f_2 be functions from *Boolean*^{3} into *Boolean*. Suppose for all elements x, y, z of *Boolean* holds $f_1(\langle x,y,z\rangle) = \mathcal{F}(x,y,z)$ and for all elements x, y, z of *Boolean* holds $f_2(\langle x,y,z\rangle) = \mathcal{F}(x,y,z)$. Then $f_1 = f_2$

for all values of the parameter.

The scheme *3AryBooleDef* deals with a ternary functor $\mathcal{F}$ yielding an element of *Boolean*, and states that:

(i) There exists a function f from *Boolean*^{3} into *Boolean* such that for all elements x, y, z of *Boolean* holds $f(\langle x,y,z\rangle) = \mathcal{F}(x,y,z)$, and

(ii) for all functions f_1, f_2 from *Boolean*^{3} into *Boolean* such that for all elements x, y, z of *Boolean* holds $f_1(\langle x,y,z\rangle) = \mathcal{F}(x,y,z)$ and for all elements x, y, z of *Boolean* holds $f_2(\langle x,y,z\rangle) = \mathcal{F}(x,y,z)$ holds $f_1 = f_2$

for all values of the parameter.

The function xor from *Boolean*^{2} into *Boolean* is defined by:

(Def. 4) For all elements x, y of *Boolean* holds $\mathrm{xor}(\langle x,y\rangle) = x \oplus y$.

The function or from *Boolean*^{2} into *Boolean* is defined as follows:

(Def. 5) For all elements x, y of *Boolean* holds $\mathrm{or}(\langle x,y\rangle) = x \vee y$.

The function & from *Boolean*^{2} into *Boolean* is defined by:

(Def. 6) For all elements x, y of *Boolean* holds $\&(\langle x,y\rangle) = x \wedge y$.

The function or_3 from *Boolean*^{3} into *Boolean* is defined as follows:

(Def. 7) For all elements x, y, z of *Boolean* holds $\mathrm{or}_3(\langle x,y,z\rangle) = x \vee y \vee z$.

Let x be a set. Then $\langle x\rangle$ is a finite sequence with length 1. Let y be a set. Then $\langle x,y\rangle$ is a finite sequence with length 2. Let z be a set. Then $\langle x,y,z\rangle$ is a finite sequence with length 3.

Let n, m be natural numbers, let p be a finite sequence with length n, and let q be a finite sequence with length m. Then $p \frown q$ is a finite sequence with length $n+m$.

3. Signatures with One Operation

One can prove the following proposition

(10) Let S be a circuit-like non void non empty many sorted signature, A be a non-empty circuit of S, s be a state of A, and g be a gate of S. Then $(\mathrm{Following}(s))(\text{the result sort of } g) = (\mathrm{Den}(g, A))(s \cdot \mathrm{Arity}(g))$.

Let S be a non void circuit-like non empty many sorted signature, let A be a non-empty circuit of S, let s be a state of A, and let n be a natural number. The functor Following(s,n) yields a state of A and is defined by the condition (Def. 8).

(Def. 8) There exists a function f from $\mathbb{N}$ into $\prod$(the sorts of A) such that Following$(s,n) = f(n)$ and $f(0) = s$ and for every natural number n holds $f(n+1) =$ Following$(f(n))$.

One can prove the following propositions:

(11) Let S be a circuit-like non void non empty many sorted signature, A be a non-empty circuit of S, and s be a state of A. Then Following$(s,0) = s$.

(12) Let S be a circuit-like non void non empty many sorted signature, A be a non-empty circuit of S, s be a state of A, and n be a natural number. Then Following$(s,n+1) =$ Following(Following(s,n)).

(13) Let S be a circuit-like non void non empty many sorted signature, A be a non-empty circuit of S, s be a state of A, and n, m be natural numbers. Then Following$(s,n+m) =$ Following(Following$(s,n),m$).

(14) Let S be a non void circuit-like non empty many sorted signature, A be a non-empty circuit of S, and s be a state of A. Then Following$(s,1) =$ Following(s).

(15) Let S be a non void circuit-like non empty many sorted signature, A be a non-empty circuit of S, and s be a state of A. Then Following$(s,2) =$ Following(Following(s)).

(16) Let S be a circuit-like non void non empty many sorted signature, A be a non-empty circuit of S, s be a state of A, and n be a natural number. Then Following$(s,n+1) =$ Following(Following$(s),n$).

Let S be a non void circuit-like non empty many sorted signature, let A be a non-empty circuit of S, let s be a state of A, and let x be a set. We say that s is stable at x if and only if:

(Def. 9) For every natural number n holds (Following$(s,n))(x) = s(x)$.

The following propositions are true:

(17) Let S be a non void circuit-like non empty many sorted signature, A be a non-empty circuit of S, s be a state of A, and x be a set. If s is stable at x, then for every natural number n holds Following(s,n) is stable at x.

(18) Let S be a non void circuit-like non empty many sorted signature, A be a non-empty circuit of S, s be a state of A, and x be a set. If $x \in$ InputVertices(S), then s is stable at x.

(19) Let S be a non void circuit-like non empty many sorted signature, A be a non-empty circuit of S, s be a state of A, and g be a gate of S. Suppose that for every set x such that $x \in$ rng Arity(g) holds s is stable at x. Then Following(s) is stable at the result sort of g.

4. Unsplit Condition

Next we state a number of propositions:

(20) Let S_1, S_2 be non empty many sorted signatures and v be a vertex of S_1. Then $v \in$ the carrier of $S_1 {+\cdot} S_2$ and $v \in$ the carrier of $S_2 {+\cdot} S_1$.

(21) Let S_1, S_2 be unsplit non empty many sorted signatures with arity held in gates and x be a set. If $x \in$ InnerVertices(S_1), then $x \in$ InnerVertices$(S_1 {+\cdot} S_2)$ and $x \in$ InnerVertices$(S_2 {+\cdot} S_1)$.

(22) For all non empty many sorted signatures S_1, S_2 and for every set x such that $x \in$ InnerVertices(S_2) holds $x \in$ InnerVertices$(S_1 {+\cdot} S_2)$.

(23) For all unsplit non empty many sorted signatures S_1, S_2 with arity held in gates holds $S_1 {+\cdot} S_2 = S_2 {+\cdot} S_1$.

(24) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates, A_1 be a Boolean circuit of S_1 with denotation held in gates, and A_2 be a Boolean circuit of S_2 with denotation held in gates. Then $A_1 {+\cdot} A_2 = A_2 {+\cdot} A_1$.

(25) Let S_1, S_2, S_3 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates, A_1 be a Boolean circuit of S_1, A_2 be a Boolean circuit of S_2, and A_3 be a Boolean circuit of S_3. Then $(A_1 {+\cdot} A_2) {+\cdot} A_3 = A_1 {+\cdot} (A_2 {+\cdot} A_3)$.

(26) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates, A_1 be a Boolean non-empty circuit of S_1 with denotation held in gates, A_2 be a Boolean non-empty circuit of S_2 with denotation held in gates, and s be a state of $A_1 {+\cdot} A_2$. Then $s \restriction$the carrier of S_1 is a state of A_1 and $s \restriction$the carrier of S_2 is a state of A_2.

(27) For all unsplit non empty many sorted signatures S_1, S_2 with arity held in gates holds $\mathrm{InnerVertices}(S_1 {+\cdot} S_2) = \mathrm{InnerVertices}(S_1) \cup \mathrm{InnerVertices}(S_2)$.

(28) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates. Suppose $\mathrm{InnerVertices}(S_2)$ misses $\mathrm{InputVertices}(S_1)$. Let A_1 be a Boolean circuit of S_1 with denotation held in gates, A_2 be a Boolean circuit of S_2 with denotation held in gates, s be a state of $A_1 {+\cdot} A_2$, and s_1 be a state of A_1. If $s_1 = s \restriction$the carrier of S_1, then $\mathrm{Following}(s) \restriction$the carrier of $S_1 = \mathrm{Following}(s_1)$.

(29) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates. Suppose $\mathrm{InnerVertices}(S_1)$ misses $\mathrm{InputVertices}(S_2)$. Let A_1 be a Boolean circuit of S_1 with denotation held in gates, A_2 be a Boolean circuit of S_2 with denotation held in gates, s be a state of $A_1 {+\cdot} A_2$, and s_2 be a state of A_2. If $s_2 = s \restriction$the carrier of S_2, then $\mathrm{Following}(s) \restriction$the carrier of $S_2 = \mathrm{Following}(s_2)$.

(30) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates. Suppose $\mathrm{InnerVertices}(S_2)$ misses $\mathrm{InputVertices}(S_1)$. Let A_1 be a Boolean circuit of S_1 with denotation held in gates, A_2 be a Boolean circuit of S_2 with denotation held in gates, s be a state of $A_1 {+\cdot} A_2$, and s_1 be a state of A_1. Suppose $s_1 = s \restriction$the carrier of S_1. Let n be a natural number. Then $\mathrm{Following}(s,n) \restriction$the carrier of $S_1 = \mathrm{Following}(s_1,n)$.

(31) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates. Suppose $\mathrm{InnerVertices}(S_1)$ misses $\mathrm{InputVertices}(S_2)$. Let A_1 be a Boolean circuit of S_1 with denotation held in gates, A_2 be a Boolean circuit of S_2 with denotation held in gates, s be a state of $A_1 {+\cdot} A_2$, and s_2 be a state of A_2. Suppose $s_2 = s \restriction$the carrier of S_2. Let n be a natural number. Then $\mathrm{Following}(s,n) \restriction$the carrier of $S_2 = \mathrm{Following}(s_2,n)$.

(32) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates. Suppose $\mathrm{InnerVertices}(S_2)$ misses $\mathrm{InputVertices}(S_1)$. Let A_1 be a Boolean circuit of S_1 with denotation held in gates, A_2 be a Boolean circuit of S_2 with denotation held in gates, s be a state of $A_1 {+\cdot} A_2$, and s_1 be a state of A_1. Suppose $s_1 = s \restriction$the carrier of S_1. Let v be a set. Suppose $v \in$ the carrier of S_1. Let n be a natural number. Then $(\mathrm{Following}(s,n))(v) = (\mathrm{Following}(s_1,n))(v)$.

(33) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates. Suppose $\mathrm{InnerVertices}(S_1)$ misses $\mathrm{InputVertices}(S_2)$. Let A_1 be a Boolean circuit of S_1 with denotation held in gates, A_2 be a Boolean circuit of S_2 with denotation held in gates, s be a state of $A_1 {+\cdot} A_2$, and s_2 be a state of A_2. Suppose $s_2 = s \restriction$the carrier of S_2. Let v be a set. Suppose $v \in$ the carrier of S_2. Let n be a natural number. Then $(\mathrm{Following}(s,n))(v) = (\mathrm{Following}(s_2,n))(v)$.

Let S be a non void non empty many sorted signature with denotation held in gates and let g be a gate of S. One can verify that $g_{\mathbf{2}}$ is function-like and relation-like.

The following four propositions are true:

(34) Let S be a circuit-like non void non empty many sorted signature with denotation held in gates and A be a non-empty circuit of S. Suppose A has denotation held in gates. Let s be a state of A and g be a gate of S. Then $(\mathrm{Following}(s))(\text{the result sort of } g) = g_{\mathbf{2}}(s \cdot \mathrm{Arity}(g))$.

(35) Let S be an unsplit non void non empty many sorted signature with arity held in gates and Boolean denotation held in gates, A be a Boolean non-empty circuit of S with denotation held in gates, s be a state of A, p be a finite sequence, and f be a function. If $\langle p, f \rangle \in$ the operation symbols of S, then $(\mathrm{Following}(s))(\langle p, f \rangle) = f(s \cdot p)$.

(36) Let S be an unsplit non void non empty many sorted signature with arity held in gates and Boolean denotation held in gates, A be a Boolean non-empty circuit of S with denotation held in gates, s be a state of A, p be a finite sequence, and f be a function. Suppose $\langle p, f \rangle \in$ the operation symbols of S and for every set x such that $x \in \mathrm{rng}\, p$ holds s is stable at x. Then $\mathrm{Following}(s)$ is stable at $\langle p, f \rangle$.

(37) For every unsplit non empty many sorted signature S holds $\mathrm{InnerVertices}(S) =$ the operation symbols of S.

5. One Gate Circuits

Next we state a number of propositions:

(38) For every set f and for every finite sequence p holds $\mathrm{InnerVertices}(\mathrm{1GateCircStr}(p, f))$ is a binary relation.

(39) For every set f and for every nonpair yielding finite sequence p holds $\mathrm{InputVertices}(\mathrm{1GateCircStr}(p, f))$ has no pairs.

(40) For every set f and for all sets x, y holds $\mathrm{InputVertices}(\mathrm{1GateCircStr}(\langle x, y \rangle, f)) = \{x, y\}$.

(41) For every set f and for all non pair sets x, y holds $\mathrm{InputVertices}(\mathrm{1GateCircStr}(\langle x, y \rangle, f))$ has no pairs.

(42) For every set f and for all sets x, y, z holds $\mathrm{InputVertices}(\mathrm{1GateCircStr}(\langle x, y, z \rangle, f)) = \{x, y, z\}$.

(43) Let x, y, f be sets. Then $x \in$ the carrier of $\mathrm{1GateCircStr}(\langle x, y \rangle, f)$ and $y \in$ the carrier of $\mathrm{1GateCircStr}(\langle x, y \rangle, f)$ and $\langle \langle x, y \rangle, f \rangle \in$ the carrier of $\mathrm{1GateCircStr}(\langle x, y \rangle, f)$.

(44) Let x, y, z, f be sets. Then $x \in$ the carrier of $\mathrm{1GateCircStr}(\langle x, y, z \rangle, f)$ and $y \in$ the carrier of $\mathrm{1GateCircStr}(\langle x, y, z \rangle, f)$ and $z \in$ the carrier of $\mathrm{1GateCircStr}(\langle x, y, z \rangle, f)$.

(45) Let f, x be sets and p be a finite sequence. Then $x \in$ the carrier of $\mathrm{1GateCircStr}(p, f, x)$ and for every set y such that $y \in \mathrm{rng}\, p$ holds $y \in$ the carrier of $\mathrm{1GateCircStr}(p, f, x)$.

(46) For all sets f, x and for every finite sequence p holds $\mathrm{1GateCircStr}(p, f, x)$ is circuit-like and has arity held in gates.

(47) For every finite sequence p and for every set f holds $\langle p, f \rangle \in \mathrm{InnerVertices}(\mathrm{1GateCircStr}(p, f))$.

Let x, y be sets and let f be a function from $Boolean^2$ into $Boolean$. The functor $\mathrm{1GateCircuit}(x, y, f)$ yields a Boolean strict circuit of $\mathrm{1GateCircStr}(\langle x, y \rangle, f)$ with denotation held in gates and is defined by:

(Def. 10) $\mathrm{1GateCircuit}(x, y, f) = \mathrm{1GateCircuit}(\langle x, y \rangle, f)$.

We use the following convention: x, y, z, c denote sets and f denotes a function from $\mathit{Boolean}^2$ into *Boolean*.

Next we state four propositions:

(48) Let X be a finite non empty set, f be a function from X^2 into X, and s be a state of $1\mathrm{GateCircuit}(\langle x,y\rangle,f)$. Then $(\mathrm{Following}(s))(\langle\!\langle\langle x,y\rangle, f\rangle\!\rangle) = f(\langle s(x),s(y)\rangle)$ and $(\mathrm{Following}(s))(x) = s(x)$ and $(\mathrm{Following}(s))(y) = s(y)$.

(49) Let X be a finite non empty set, f be a function from X^2 into X, and s be a state of $1\mathrm{GateCircuit}(\langle x,y\rangle,f)$. Then $\mathrm{Following}(s)$ is stable.

(50) For every state s of $1\mathrm{GateCircuit}(x,y,f)$ holds $(\mathrm{Following}(s))(\langle\!\langle\langle x,y\rangle, f\rangle\!\rangle) = f(\langle s(x),s(y)\rangle)$ and $(\mathrm{Following}(s))(x) = s(x)$ and $(\mathrm{Following}(s))(y) = s(y)$.

(51) For every state s of $1\mathrm{GateCircuit}(x,y,f)$ holds $\mathrm{Following}(s)$ is stable.

Let x, y, z be sets and let f be a function from $\mathit{Boolean}^3$ into *Boolean*. The functor $1\mathrm{GateCircuit}(x,y,z,f)$ yields a Boolean strict circuit of $1\mathrm{GateCircStr}(\langle x,y,z\rangle,f)$ with denotation held in gates and is defined as follows:

(Def. 11) $1\mathrm{GateCircuit}(x,y,z,f) = 1\mathrm{GateCircuit}(\langle x,y,z\rangle,f)$.

Next we state four propositions:

(52) Let X be a finite non empty set, f be a function from X^3 into X, and s be a state of $1\mathrm{GateCircuit}(\langle x,y,z\rangle,f)$. Then $(\mathrm{Following}(s))(\langle\!\langle\langle x,y,z\rangle, f\rangle\!\rangle) = f(\langle s(x),s(y),s(z)\rangle)$ and $(\mathrm{Following}(s))(x) = s(x)$ and $(\mathrm{Following}(s))(y) = s(y)$ and $(\mathrm{Following}(s))(z) = s(z)$.

(53) Let X be a finite non empty set, f be a function from X^3 into X, and s be a state of $1\mathrm{GateCircuit}(\langle x,y,z\rangle,f)$. Then $\mathrm{Following}(s)$ is stable.

(54) Let f be a function from $\mathit{Boolean}^3$ into *Boolean* and s be a state of $1\mathrm{GateCircuit}(x,y,z,f)$. Then $(\mathrm{Following}(s))(\langle\!\langle\langle x,y,z\rangle, f\rangle\!\rangle) = f(\langle s(x),s(y),s(z)\rangle)$ and $(\mathrm{Following}(s))(x) = s(x)$ and $(\mathrm{Following}(s))(y) = s(y)$ and $(\mathrm{Following}(s))(z) = s(z)$.

(55) For every function f from $\mathit{Boolean}^3$ into *Boolean* and for every state s of $1\mathrm{GateCircuit}(x,y,z,f)$ holds $\mathrm{Following}(s)$ is stable.

6. Boolean Circuits

Let x, y, c be sets and let f be a function from $\mathit{Boolean}^2$ into *Boolean*. The functor $2\mathrm{GatesCircStr}(x,y,c,f)$ yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

(Def. 12) $2\mathrm{GatesCircStr}(x,y,c,f) = 1\mathrm{GateCircStr}(\langle x,y\rangle,f) +\cdot\, 1\mathrm{GateCircStr}(\langle\langle\!\langle\langle x,y\rangle, f\rangle\!\rangle,c\rangle,f)$.

Let x, y, c be sets and let f be a function from $\mathit{Boolean}^2$ into *Boolean*. The functor $2\mathrm{GatesCircOutput}(x,y,c,f)$ yields an element of $\mathrm{InnerVertices}(2\mathrm{GatesCircStr}(x,y,c,f))$ and is defined by:

(Def. 13) $2\mathrm{GatesCircOutput}(x,y,c,f) = \langle\!\langle\langle\langle\!\langle\langle x,y\rangle, f\rangle\!\rangle,c\rangle, f\rangle\!\rangle$.

Let x, y, c be sets and let f be a function from $\mathit{Boolean}^2$ into *Boolean*. Note that $2\mathrm{GatesCircOutput}(x,y,c,f)$ is pair.

Next we state two propositions:

(56) $\mathrm{InnerVertices}(2\mathrm{GatesCircStr}(x,y,c,f)) = \{\langle\!\langle\langle x,y\rangle, f\rangle\!\rangle, 2\mathrm{GatesCircOutput}(x,y,c,f)\}$.

(57) If $c \neq \langle\!\langle\langle x,y\rangle, f\rangle\!\rangle$, then $\mathrm{InnerVertices}(2\mathrm{GatesCircStr}(x,y,c,f)) = \{x,y,c\}$.

Let x, y, c be sets and let f be a function from *Boolean*^{2} into *Boolean*. The functor $\mathrm{2GatesCircuit}(x,y,c,f)$ yielding a strict Boolean circuit of $\mathrm{2GatesCircStr}(x,y,c,f)$ with denotation held in gates is defined as follows:

(Def. 14) $\mathrm{2GatesCircuit}(x,y,c,f) = \mathrm{1GateCircuit}(x,y,f) +\cdot \mathrm{1GateCircuit}(\langle\!\langle\langle x,y\rangle, f\rangle\!\rangle, c, f)$.

One can prove the following four propositions:

(58) $\mathrm{InnerVertices}(\mathrm{2GatesCircStr}(x,y,c,f))$ is a binary relation.

(59) For all non pair sets x, y, c holds $\mathrm{InputVertices}(\mathrm{2GatesCircStr}(x,y,c,f))$ has no pairs.

(60) $x \in$ the carrier of $\mathrm{2GatesCircStr}(x,y,c,f)$ and $y \in$ the carrier of $\mathrm{2GatesCircStr}(x,y,c,f)$ and $c \in$ the carrier of $\mathrm{2GatesCircStr}(x,y,c,f)$.

(61) $\langle\!\langle\langle x,y\rangle, f\rangle\!\rangle \in$ the carrier of $\mathrm{2GatesCircStr}(x,y,c,f)$ and $\langle\!\langle\langle\langle\!\langle\langle x,y\rangle, f\rangle\!\rangle, c\rangle, f\rangle\!\rangle \in$ the carrier of $\mathrm{2GatesCircStr}(x,y,c,f)$.

Let S be an unsplit non void non empty many sorted signature, let A be a Boolean circuit of S, let s be a state of A, and let v be a vertex of S. Then $s(v)$ is an element of *Boolean*.

In the sequel s is a state of $\mathrm{2GatesCircuit}(x,y,c,f)$.

We now state several propositions:

(62) Suppose $c \neq \langle\!\langle\langle x,y\rangle, f\rangle\!\rangle$. Then $(\mathrm{Following}(s,2))(\mathrm{2GatesCircOutput}(x,y,c,f)) = f(\langle f(\langle s(x), s(y)\rangle), s(c)\rangle)$ and $(\mathrm{Following}(s,2))(\langle\!\langle\langle x,y\rangle, f\rangle\!\rangle) = f(\langle s(x), s(y)\rangle)$ and $(\mathrm{Following}(s,2))(x) = s(x)$ and $(\mathrm{Following}(s,2))(y) = s(y)$ and $(\mathrm{Following}(s,2))(c) = s(c)$.

(63) If $c \neq \langle\!\langle\langle x,y\rangle, f\rangle\!\rangle$, then $\mathrm{Following}(s,2)$ is stable.

(64) Suppose $c \neq \langle\!\langle\langle x,y\rangle, \mathrm{xor}\rangle\!\rangle$. Let s be a state of $\mathrm{2GatesCircuit}(x,y,c,\mathrm{xor})$ and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(c)$, then $(\mathrm{Following}(s,2))(\mathrm{2GatesCircOutput}(x,y,c,\mathrm{xor})) = a_1 \oplus a_2 \oplus a_3$.

(65) Suppose $c \neq \langle\!\langle\langle x,y\rangle, \mathrm{or}\rangle\!\rangle$. Let s be a state of $\mathrm{2GatesCircuit}(x,y,c,\mathrm{or})$ and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(c)$, then $(\mathrm{Following}(s,2))(\mathrm{2GatesCircOutput}(x,y,c,\mathrm{or})) = a_1 \vee a_2 \vee a_3$.

(66) Suppose $c \neq \langle\!\langle\langle x,y\rangle, \&\rangle\!\rangle$. Let s be a state of $\mathrm{2GatesCircuit}(x,y,c,\&)$ and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(c)$, then $(\mathrm{Following}(s,2))(\mathrm{2GatesCircOutput}(x,y,c,\&)) = a_1 \wedge a_2 \wedge a_3$.

7. One Bit Adder

Let x, y, c be sets. The functor $\mathrm{BitAdderOutput}(x,y,c)$ yielding an element of $\mathrm{InnerVertices}(\mathrm{2GatesCircStr}(x,y,c,\mathrm{xor}))$ is defined by:

(Def. 15) $\mathrm{BitAdderOutput}(x,y,c) = \mathrm{2GatesCircOutput}(x,y,c,\mathrm{xor})$.

Let x, y, c be sets. The functor $\mathrm{BitAdderCirc}(x,y,c)$ yielding a strict Boolean circuit of $\mathrm{2GatesCircStr}(x,y,c,\mathrm{xor})$ with denotation held in gates is defined by:

(Def. 16) $\mathrm{BitAdderCirc}(x,y,c) = \mathrm{2GatesCircuit}(x,y,c,\mathrm{xor})$.

Let x, y, c be sets. The functor $\mathrm{MajorityIStr}(x,y,c)$ yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined as follows:

(Def. 17) $\mathrm{MajorityIStr}(x,y,c) = \mathrm{1GateCircStr}(\langle x,y\rangle, \&) +\cdot \mathrm{1GateCircStr}(\langle y,c\rangle, \&) +\cdot \mathrm{1GateCircStr}(\langle c, x\rangle, \&)$.

Let x, y, c be sets. The functor $\mathrm{MajorityStr}(x,y,c)$ yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined as follows:

(Def. 18) $\mathrm{MajorityStr}(x,y,c) = \mathrm{MajorityIStr}(x,y,c) +\cdot \mathrm{1GateCircStr}(\langle \langle \langle x,y \rangle, \& \rangle, \langle \langle y,c \rangle, \& \rangle, \langle \langle c,x \rangle, \& \rangle \rangle, \mathrm{or}_3)$.

Let x, y, c be sets. The functor $\mathrm{MajorityICirc}(x,y,c)$ yields a strict Boolean circuit of $\mathrm{MajorityIStr}(x,y,c)$ with denotation held in gates and is defined as follows:

(Def. 19) $\mathrm{MajorityICirc}(x,y,c) = \mathrm{1GateCircuit}(x,y,\&) +\cdot \mathrm{1GateCircuit}(y,c,\&) +\cdot \mathrm{1GateCircuit}(c,x,\&)$.

One can prove the following propositions:

(67) $\mathrm{InnerVertices}(\mathrm{MajorityStr}(x,y,c))$ is a binary relation.

(68) For all non pair sets x, y, c holds $\mathrm{InputVertices}(\mathrm{MajorityStr}(x,y,c))$ has no pairs.

(69) For every state s of $\mathrm{MajorityICirc}(x,y,c)$ and for all elements a, b of *Boolean* such that $a = s(x)$ and $b = s(y)$ holds $(\mathrm{Following}(s))(\langle \langle x,y \rangle, \& \rangle) = a \wedge b$.

(70) For every state s of $\mathrm{MajorityICirc}(x,y,c)$ and for all elements a, b of *Boolean* such that $a = s(y)$ and $b = s(c)$ holds $(\mathrm{Following}(s))(\langle \langle y,c \rangle, \& \rangle) = a \wedge b$.

(71) For every state s of $\mathrm{MajorityICirc}(x,y,c)$ and for all elements a, b of *Boolean* such that $a = s(c)$ and $b = s(x)$ holds $(\mathrm{Following}(s))(\langle \langle c,x \rangle, \& \rangle) = a \wedge b$.

Let x, y, c be sets. The functor $\mathrm{MajorityOutput}(x,y,c)$ yields an element of $\mathrm{InnerVertices}(\mathrm{MajorityStr}(x,y,c))$ and is defined by:

(Def. 20) $\mathrm{MajorityOutput}(x,y,c) = \langle \langle \langle \langle x,y \rangle, \& \rangle, \langle \langle y,c \rangle, \& \rangle, \langle \langle c,x \rangle, \& \rangle \rangle, \mathrm{or}_3 \rangle$.

Let x, y, c be sets. The functor $\mathrm{MajorityCirc}(x,y,c)$ yields a strict Boolean circuit of $\mathrm{MajorityStr}(x,y,c)$ with denotation held in gates and is defined as follows:

(Def. 21) $\mathrm{MajorityCirc}(x,y,c) = \mathrm{MajorityICirc}(x,y,c) +\cdot \mathrm{1GateCircuit}(\langle \langle x,y \rangle, \& \rangle, \langle \langle y,c \rangle, \& \rangle, \langle \langle c,x \rangle, \& \rangle, \mathrm{or}_3)$.

We now state a number of propositions:

(72) $x \in$ the carrier of $\mathrm{MajorityStr}(x,y,c)$ and $y \in$ the carrier of $\mathrm{MajorityStr}(x,y,c)$ and $c \in$ the carrier of $\mathrm{MajorityStr}(x,y,c)$.

(73) $\langle \langle x,y \rangle, \& \rangle \in \mathrm{InnerVertices}(\mathrm{MajorityStr}(x,y,c))$ and $\langle \langle y,c \rangle, \& \rangle \in \mathrm{InnerVertices}(\mathrm{MajorityStr}(x,y,c))$ and $\langle \langle c,x \rangle, \& \rangle \in \mathrm{InnerVertices}(\mathrm{MajorityStr}(x,y,c))$.

(74) For all non pair sets x, y, c holds $x \in \mathrm{InputVertices}(\mathrm{MajorityStr}(x,y,c))$ and $y \in \mathrm{InputVertices}(\mathrm{MajorityStr}(x,y,c))$ and $c \in \mathrm{InputVertices}(\mathrm{MajorityStr}(x,y,c))$.

(75) For all non pair sets x, y, c holds $\mathrm{InputVertices}(\mathrm{MajorityStr}(x,y,c)) = \{x,y,c\}$ and $\mathrm{InnerVertices}(\mathrm{MajorityStr}(x,y,c)) = \{\langle \langle x,y \rangle, \& \rangle, \langle \langle y,c \rangle, \& \rangle, \langle \langle c,x \rangle, \& \rangle\} \cup \{\mathrm{MajorityOutput}(x,y,c)\}$.

(76) Let x, y, c be non pair sets, s be a state of $\mathrm{MajorityCirc}(x,y,c)$, and a_1, a_2 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$, then $(\mathrm{Following}(s))(\langle \langle x,y \rangle, \& \rangle) = a_1 \wedge a_2$.

(77) Let x, y, c be non pair sets, s be a state of $\mathrm{MajorityCirc}(x,y,c)$, and a_2, a_3 be elements of *Boolean*. If $a_2 = s(y)$ and $a_3 = s(c)$, then $(\mathrm{Following}(s))(\langle \langle y,c \rangle, \& \rangle) = a_2 \wedge a_3$.

(78) Let x, y, c be non pair sets, s be a state of $\mathrm{MajorityCirc}(x,y,c)$, and a_1, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_3 = s(c)$, then $(\mathrm{Following}(s))(\langle \langle c,x \rangle, \& \rangle) = a_3 \wedge a_1$.

(79) Let x, y, c be non pair sets, s be a state of $\mathrm{MajorityCirc}(x,y,c)$, and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(\langle \langle x,y \rangle, \& \rangle)$ and $a_2 = s(\langle \langle y,c \rangle, \& \rangle)$ and $a_3 = s(\langle \langle c,x \rangle, \& \rangle)$, then $(\mathrm{Following}(s))(\mathrm{MajorityOutput}(x,y,c)) = a_1 \vee a_2 \vee a_3$.

(80) Let x, y, c be non pair sets, s be a state of $\mathrm{MajorityCirc}(x,y,c)$, and a_1, a_2 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$, then $(\mathrm{Following}(s,2))(\langle \langle x,y \rangle, \& \rangle) = a_1 \wedge a_2$.

(81) Let x, y, c be non pair sets, s be a state of MajorityCirc(x,y,c), and a_2, a_3 be elements of *Boolean*. If $a_2 = s(y)$ and $a_3 = s(c)$, then $(\text{Following}(s,2))(\langle\langle y,c\rangle, \&\rangle) = a_2 \wedge a_3$.

(82) Let x, y, c be non pair sets, s be a state of MajorityCirc(x,y,c), and a_1, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_3 = s(c)$, then $(\text{Following}(s,2))(\langle\langle c,x\rangle, \&\rangle) = a_3 \wedge a_1$.

(83) Let x, y, c be non pair sets, s be a state of MajorityCirc(x,y,c), and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(c)$, then $(\text{Following}(s,2))(\text{MajorityOutput}(x,y,c)) = a_1 \wedge a_2 \vee a_2 \wedge a_3 \vee a_3 \wedge a_1$.

(84) For all non pair sets x, y, c and for every state s of MajorityCirc(x,y,c) holds Following$(s,2)$ is stable.

Let x, y, c be sets. The functor BitAdderWithOverflowStr(x,y,c) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined as follows:

(Def. 22) BitAdderWithOverflowStr$(x,y,c) =$ 2GatesCircStr$(x,y,c,\text{xor}) +\cdot$ MajorityStr(x,y,c).

One can prove the following three propositions:

(85) For all non pair sets x, y, c holds InputVertices(BitAdderWithOverflowStr$(x,y,c)) = \{x,y,c\}$.

(86) For all non pair sets x, y, c holds InnerVertices(BitAdderWithOverflowStr$(x,y,c)) = \{\langle\langle x, y\rangle, \text{xor}\rangle, \text{2GatesCircOutput}(x,y,c,\text{xor})\} \cup \{\langle\langle x,y\rangle, \&\rangle, \langle\langle y,c\rangle, \&\rangle, \langle\langle c,x\rangle, \&\rangle\} \cup \{\text{MajorityOutput}(x,y,c)\}$.

(87) Let S be a non empty many sorted signature. Suppose $S =$ BitAdderWithOverflowStr(x,y,c). Then $x \in$ the carrier of S and $y \in$ the carrier of S and $c \in$ the carrier of S.

Let x, y, c be sets. The functor BitAdderWithOverflowCirc(x,y,c) yielding a strict Boolean circuit of BitAdderWithOverflowStr(x,y,c) with denotation held in gates is defined by:

(Def. 23) BitAdderWithOverflowCirc$(x,y,c) =$ BitAdderCirc$(x,y,c) +\cdot$ MajorityCirc(x,y,c).

One can prove the following propositions:

(88) InnerVertices(BitAdderWithOverflowStr(x,y,c)) is a binary relation.

(89) For all non pair sets x, y, c holds InputVertices(BitAdderWithOverflowStr(x,y,c)) has no pairs.

(90) BitAdderOutput$(x,y,c) \in$ InnerVertices(BitAdderWithOverflowStr(x,y,c)) and MajorityOutput$(x,y,c) \in$ InnerVertices(BitAdderWithOverflowStr(x,y,c)).

(91) Let x, y, c be non pair sets, s be a state of BitAdderWithOverflowCirc(x,y,c), and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(c)$. Then $(\text{Following}(s,2))(\text{BitAdderOutput}(x,y,c)) = a_1 \oplus a_2 \oplus a_3$ and $(\text{Following}(s,2))(\text{MajorityOutput}(x,y,c)) = a_1 \wedge a_2 \vee a_2 \wedge a_3 \vee a_3 \wedge a_1$.

(92) For all non pair sets x, y, c and for every state s of BitAdderWithOverflowCirc(x,y,c) holds Following$(s,2)$ is stable.

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Received August 10, 1995

Published January 2, 2004