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Category Ens

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Summary. If V is any non-empty set of sets, we define $\mathbf{Ens}_V$ to be the category with the objects of all sets $X \in V$, morphisms of all mappings from X into Y, with the usual composition of mappings. By a mapping we mean a triple $\langle X, Y, f \rangle$ where f is a function from X into Y. The notations and concepts included corresponds to that presented in [12], [10]. We also introduce representable functors to illustrate properties of the category **Ens**.

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The articles [15], [6], [18], [16], [14], [19], [2], [3], [5], [7], [1], [17], [11], [13], [4], [8], and [9] provide the notation and terminology for this paper.

1. Mappings

In this paper V denotes a non empty set and A, B denote elements of V.

Let us consider V. The functor $\mathrm{Funcs}\,V$ yielding a set is defined as follows:

(Def. 1) $\mathrm{Funcs}\,V = \bigcup\{B^A\}$.

Let us consider V. Note that $\mathrm{Funcs}\,V$ is functional and non empty.

We now state three propositions:

(1) For every set f holds $f \in \mathrm{Funcs}\,V$ iff there exist A, B such that if $B = \emptyset$, then $A = \emptyset$ and f is a function from A into B.

(2) $B^A \subseteq \mathrm{Funcs}\,V$.

(3) For every non empty subset W of V holds $\mathrm{Funcs}\,W \subseteq \mathrm{Funcs}\,V$.

In the sequel f denotes an element of $\mathrm{Funcs}\,V$.

Let us consider V. The functor $\mathrm{Maps}\,V$ yields a set and is defined as follows:

(Def. 2) $\mathrm{Maps}\,V = \{\langle\langle A, B\rangle, f\rangle : (B = \emptyset \Rightarrow A = \emptyset) \wedge f$ is a function from A into $B\}$.

Let us consider V. One can verify that $\mathrm{Maps}\,V$ is non empty.

In the sequel m, m_1, m_2, m_3 denote elements of $\mathrm{Maps}\,V$.

We now state four propositions:

(4) There exist f, A, B such that $m = \langle\langle A, B\rangle, f\rangle$ and if $B = \emptyset$, then $A = \emptyset$ and f is a function from A into B.

(5) For every function f from A into B such that if $B = \emptyset$, then $A = \emptyset$ holds $\langle\langle A, B\rangle, f\rangle \in \mathrm{Maps}\,V$.

(6) $\mathrm{Maps}\,V \subseteq [:[:V, V:], \mathrm{Funcs}\,V:]$.

(7) For every non empty subset W of V holds $\mathrm{Maps}\,W \subseteq \mathrm{Maps}\,V$.

Let V be a non empty set and let m be an element of $\mathrm{Maps}\,V$. Observe that $m_{\mathbf{2}}$ is function-like and relation-like.

Let us consider V, m. The functor $\mathrm{dom}\,m$ yielding an element of V is defined by:

(Def. 4)[1] $\mathrm{dom}\,m = (m_{\mathbf{1}})_{\mathbf{1}}$.

The functor $\mathrm{cod}\,m$ yields an element of V and is defined by:

(Def. 5) $\mathrm{cod}\,m = (m_{\mathbf{1}})_{\mathbf{2}}$.

We now state three propositions:

(8) $m = \langle\langle \mathrm{dom}\,m, \mathrm{cod}\,m\rangle, m_{\mathbf{2}}\rangle$.

(9) $\mathrm{cod}\,m \neq \emptyset$ or $\mathrm{dom}\,m = \emptyset$ but $m_{\mathbf{2}}$ is a function from $\mathrm{dom}\,m$ into $\mathrm{cod}\,m$.

(10) Let f be a function and A, B be sets. Suppose $\langle\langle A, B\rangle, f\rangle \in \mathrm{Maps}\,V$. Then if $B = \emptyset$, then $A = \emptyset$ and f is a function from A into B.

Let us consider V, A. The functor $\mathrm{id}(A)$ yielding an element of $\mathrm{Maps}\,V$ is defined as follows:

(Def. 6) $\mathrm{id}(A) = \langle\langle A, A\rangle, \mathrm{id}_A\rangle$.

One can prove the following proposition

(11) $(\mathrm{id}(A))_{\mathbf{2}} = \mathrm{id}_A$ and $\mathrm{dom}\,\mathrm{id}(A) = A$ and $\mathrm{cod}\,\mathrm{id}(A) = A$.

Let us consider V, m_1, m_2. Let us assume that $\mathrm{cod}\,m_1 = \mathrm{dom}\,m_2$. The functor $m_2 \cdot m_1$ yields an element of $\mathrm{Maps}\,V$ and is defined by:

(Def. 7) $m_2 \cdot m_1 = \langle\langle \mathrm{dom}\,m_1, \mathrm{cod}\,m_2\rangle, (m_2)_{\mathbf{2}} \cdot (m_1)_{\mathbf{2}}\rangle$.

The following propositions are true:

(12) If $\mathrm{dom}\,m_2 = \mathrm{cod}\,m_1$, then $(m_2 \cdot m_1)_{\mathbf{2}} = (m_2)_{\mathbf{2}} \cdot (m_1)_{\mathbf{2}}$ and $\mathrm{dom}(m_2 \cdot m_1) = \mathrm{dom}\,m_1$ and $\mathrm{cod}(m_2 \cdot m_1) = \mathrm{cod}\,m_2$.

(13) If $\mathrm{dom}\,m_2 = \mathrm{cod}\,m_1$ and $\mathrm{dom}\,m_3 = \mathrm{cod}\,m_2$, then $m_3 \cdot (m_2 \cdot m_1) = (m_3 \cdot m_2) \cdot m_1$.

(14) $m \cdot \mathrm{id}(\mathrm{dom}\,m) = m$ and $\mathrm{id}(\mathrm{cod}\,m) \cdot m = m$.

Let us consider V, A, B. The functor $\mathrm{Maps}(A,B)$ yields a set and is defined as follows:

(Def. 8) $\mathrm{Maps}(A,B) = \{\langle\langle A, B\rangle, f\rangle; f$ ranges over elements of $\mathrm{Funcs}\,V$: $\langle\langle A, B\rangle, f\rangle \in \mathrm{Maps}\,V\}$.

Next we state several propositions:

(15) For every function f from A into B such that if $B = \emptyset$, then $A = \emptyset$ holds $\langle\langle A, B\rangle, f\rangle \in \mathrm{Maps}(A,B)$.

(16) If $m \in \mathrm{Maps}(A,B)$, then $m = \langle\langle A, B\rangle, m_{\mathbf{2}}\rangle$.

(17) $\mathrm{Maps}(A,B) \subseteq \mathrm{Maps}\,V$.

(18) $\mathrm{Maps}\,V = \bigcup\{\mathrm{Maps}(A,B)\}$.

(19) $m \in \mathrm{Maps}(A,B)$ iff $\mathrm{dom}\,m = A$ and $\mathrm{cod}\,m = B$.

(20) If $m \in \mathrm{Maps}(A,B)$, then $m_{\mathbf{2}} \in B^A$.

Let us consider V, m. We say that m is surjective if and only if:

(Def. 9) $\mathrm{rng}(m_{\mathbf{2}}) = \mathrm{cod}\,m$.

We introduce m is a surjection as a synonym of m is surjective.

[1] The definition (Def. 3) has been removed.

2. Category Ens

Let us consider V. The functor Dom_V yields a function from $\mathrm{Maps}\,V$ into V and is defined as follows:

(Def. 10) For every m holds $\mathrm{Dom}_V(m) = \mathrm{dom}\,m$.

The functor Cod_V yields a function from $\mathrm{Maps}\,V$ into V and is defined by:

(Def. 11) For every m holds $\mathrm{Cod}_V(m) = \mathrm{cod}\,m$.

The functor $\cdot_V$ yielding a partial function from $[:\mathrm{Maps}\,V, \mathrm{Maps}\,V:]$ to $\mathrm{Maps}\,V$ is defined as follows:

(Def. 12) For all m_2, m_1 holds $\langle m_2, m_1\rangle \in \mathrm{dom}(\cdot_V)$ iff $\mathrm{dom}\,m_2 = \mathrm{cod}\,m_1$ and for all m_2, m_1 such that $\mathrm{dom}\,m_2 = \mathrm{cod}\,m_1$ holds $\cdot_V(\langle m_2, m_1\rangle) = m_2 \cdot m_1$.

The functor Id_V yielding a function from V into $\mathrm{Maps}\,V$ is defined as follows:

(Def. 13) For every A holds $\mathrm{Id}_V(A) = \mathrm{id}(A)$.

Let us consider V. The functor $\mathbf{Ens}_V$ yielding a category structure is defined as follows:

(Def. 14) $\mathbf{Ens}_V = \langle V, \mathrm{Maps}\,V, \mathrm{Dom}_V, \mathrm{Cod}_V, \cdot_V, \mathrm{Id}_V\rangle$.

One can prove the following proposition

(21) $\langle V, \mathrm{Maps}\,V, \mathrm{Dom}_V, \mathrm{Cod}_V, \cdot_V, \mathrm{Id}_V\rangle$ is a category.

Let us consider V. Observe that $\mathbf{Ens}_V$ is strict and category-like.
In the sequel a, b denote objects of $\mathbf{Ens}_V$.
We now state the proposition

(22) A is an object of $\mathbf{Ens}_V$.

Let us consider V, A. The functor ${}^{@}A$ yields an object of $\mathbf{Ens}_V$ and is defined by:

(Def. 15) ${}^{@}A = A$.

Next we state the proposition

(23) a is an element of V.

Let us consider V, a. The functor ${}^{@}a$ yields an element of V and is defined as follows:

(Def. 16) ${}^{@}a = a$.

In the sequel f, g denote morphisms of $\mathbf{Ens}_V$.
We now state the proposition

(24) m is a morphism of $\mathbf{Ens}_V$.

Let us consider V, m. The functor ${}^{@}m$ yielding a morphism of $\mathbf{Ens}_V$ is defined by:

(Def. 17) ${}^{@}m = m$.

We now state the proposition

(25) f is an element of $\mathrm{Maps}\,V$.

Let us consider V, f. The functor ${}^{@}f$ yielding an element of $\mathrm{Maps}\,V$ is defined as follows:

(Def. 18) ${}^{@}f = f$.

The following propositions are true:

(26) $\operatorname{dom} f = \operatorname{dom}({}^{@}f)$ and $\operatorname{cod} f = \operatorname{cod}({}^{@}f)$.

(27) $\hom(a,b) = \mathrm{Maps}({}^{@}a, {}^{@}b)$.

(28) If $\operatorname{dom} g = \operatorname{cod} f$, then $g \cdot f = ({}^{@}g) \cdot ({}^{@}f)$.

(29) $\mathrm{id}_a = \mathrm{id}({}^{@}a)$.

(30) If $a = \emptyset$, then a is initial.

(31) If $\emptyset \in V$ and a is initial, then $a = \emptyset$.

(32) For every universal class W and for every object a of $\mathbf{Ens}_W$ such that a is initial holds $a = \emptyset$.

(33) If there exists a set x such that $a = \{x\}$, then a is terminal.

(34) If $V \neq \{\emptyset\}$ and a is terminal, then there exists a set x such that $a = \{x\}$.

(35) For every universal class W and for every object a of $\mathbf{Ens}_W$ such that a is terminal there exists a set x such that $a = \{x\}$.

(36) f is monic iff $({}^{@}f)_{\mathbf{2}}$ is one-to-one.

(37) If f is epi and there exists A and there exist sets x_1, x_2 such that $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$, then ${}^{@}f$ is a surjection.

(38) If ${}^{@}f$ is a surjection, then f is epi.

(39) For every universal class W and for every morphism f of $\mathbf{Ens}_W$ such that f is epi holds ${}^{@}f$ is a surjection.

(40) For every non empty subset W of V holds $\mathbf{Ens}_W$ is full subcategory of $\mathbf{Ens}_V$.

3. Representable Functors

We follow the rules: C denotes a category, a, b, c denote objects of C, and f, g, h, f', g' denote morphisms of C.

Let us consider C. The functor $\mathrm{Hom}(C)$ yielding a set is defined as follows:

(Def. 19) $\mathrm{Hom}(C) = \{\hom(a,b)\}$.

Let us consider C. Note that $\mathrm{Hom}(C)$ is non empty.

The following two propositions are true:

(41) $\hom(a,b) \in \mathrm{Hom}(C)$.

(42) If $\hom(a, \operatorname{cod} f) = \emptyset$, then $\hom(a, \operatorname{dom} f) = \emptyset$ and if $\hom(\operatorname{dom} f, a) = \emptyset$, then $\hom(\operatorname{cod} f, a) = \emptyset$.

Let us consider C, a, f. The functor $\hom(a,f)$ yields a function from $\hom(a, \operatorname{dom} f)$ into $\hom(a, \operatorname{cod} f)$ and is defined by:

(Def. 20) For every g such that $g \in \hom(a, \operatorname{dom} f)$ holds $(\hom(a,f))(g) = f \cdot g$.

The functor $\hom(f,a)$ yields a function from $\hom(\operatorname{cod} f, a)$ into $\hom(\operatorname{dom} f, a)$ and is defined by:

(Def. 21) For every g such that $g \in \hom(\operatorname{cod} f, a)$ holds $(\hom(f,a))(g) = g \cdot f$.

The following propositions are true:

(43) $\hom(a, \mathrm{id}_c) = \mathrm{id}_{\hom(a,c)}$.

(44) $\hom(\mathrm{id}_c, a) = \mathrm{id}_{\hom(c,a)}$.

(45) If $\operatorname{dom} g = \operatorname{cod} f$, then $\hom(a, g \cdot f) = \hom(a,g) \cdot \hom(a,f)$.

(46) If $\operatorname{dom} g = \operatorname{cod} f$, then $\hom(g \cdot f, a) = \hom(f, a) \cdot \hom(g, a)$.

(47) $\langle\langle \hom(a, \operatorname{dom} f), \hom(a, \operatorname{cod} f)\rangle, \hom(a, f)\rangle$ is an element of Maps Hom(C).

(48) $\langle\langle \hom(\operatorname{cod} f, a), \hom(\operatorname{dom} f, a)\rangle, \hom(f, a)\rangle$ is an element of Maps Hom(C).

Let us consider C, a. The functor $\hom(a, -)$ yielding a function from the morphisms of C into Maps Hom(C) is defined as follows:

(Def. 22) For every f holds $(\hom(a, -))(f) = \langle\langle \hom(a, \operatorname{dom} f), \hom(a, \operatorname{cod} f)\rangle, \hom(a, f)\rangle$.

The functor $\hom(-, a)$ yields a function from the morphisms of C into Maps Hom(C) and is defined as follows:

(Def. 23) For every f holds $(\hom(-, a))(f) = \langle\langle \hom(\operatorname{cod} f, a), \hom(\operatorname{dom} f, a)\rangle, \hom(f, a)\rangle$.

We now state three propositions:

(49) If $\operatorname{Hom}(C) \subseteq V$, then $\hom(a, -)$ is a functor from C to $\mathbf{Ens}_V$.

(50) If $\operatorname{Hom}(C) \subseteq V$, then $\hom(-, a)$ is a contravariant functor from C into $\mathbf{Ens}_V$.

(51) If $\hom(\operatorname{dom} f, \operatorname{cod} f') = \emptyset$, then $\hom(\operatorname{cod} f, \operatorname{dom} f') = \emptyset$.

Let us consider C, f, g. The functor $\hom(f, g)$ yields a function from $\hom(\operatorname{cod} f, \operatorname{dom} g)$ into $\hom(\operatorname{dom} f, \operatorname{cod} g)$ and is defined as follows:

(Def. 24) For every h such that $h \in \hom(\operatorname{cod} f, \operatorname{dom} g)$ holds $(\hom(f, g))(h) = g \cdot h \cdot f$.

We now state several propositions:

(52) $\langle\langle \hom(\operatorname{cod} f, \operatorname{dom} g), \hom(\operatorname{dom} f, \operatorname{cod} g)\rangle, \hom(f, g)\rangle$ is an element of Maps Hom(C).

(53) $\hom(\mathrm{id}_a, f) = \hom(a, f)$ and $\hom(f, \mathrm{id}_a) = \hom(f, a)$.

(54) $\hom(\mathrm{id}_a, \mathrm{id}_b) = \mathrm{id}_{\hom(a,b)}$.

(55) $\hom(f, g) = \hom(\operatorname{dom} f, g) \cdot \hom(f, \operatorname{dom} g)$.

(56) If $\operatorname{cod} g = \operatorname{dom} f$ and $\operatorname{dom} g' = \operatorname{cod} f'$, then $\hom(f \cdot g, g' \cdot f') = \hom(g, g') \cdot \hom(f, f')$.

Let us consider C. The functor $\hom_C(-, -)$ yielding a function from the morphisms of $[: C, C :]$ into Maps Hom(C) is defined as follows:

(Def. 25) For all f, g holds $(\hom_C(-, -))(\langle f, g\rangle) = \langle\langle \hom(\operatorname{cod} f, \operatorname{dom} g), \hom(\operatorname{dom} f, \operatorname{cod} g)\rangle, \hom(f, g)\rangle$.

Next we state two propositions:

(57) $\hom(a, -) = (\operatorname{curry}(\hom_C(-, -)))(\mathrm{id}_a)$ and $\hom(-, a) = (\operatorname{curry}'(\hom_C(-, -)))(\mathrm{id}_a)$.

(58) If $\operatorname{Hom}(C) \subseteq V$, then $\hom_C(-, -)$ is a functor from $[: C^{\mathrm{op}}, C :]$ to $\mathbf{Ens}_V$.

Let us consider V, C, a. Let us assume that $\operatorname{Hom}(C) \subseteq V$. The functor $\hom_V(a, -)$ yielding a functor from C to $\mathbf{Ens}_V$ is defined as follows:

(Def. 26) $\hom_V(a, -) = \hom(a, -)$.

The functor $\hom_V(-, a)$ yields a contravariant functor from C into $\mathbf{Ens}_V$ and is defined by:

(Def. 27) $\hom_V(-, a) = \hom(-, a)$.

Let us consider V, C. Let us assume that $\operatorname{Hom}(C) \subseteq V$. The functor $\hom^C_V(-, -)$ yielding a functor from $[: C^{\mathrm{op}}, C :]$ to $\mathbf{Ens}_V$ is defined as follows:

(Def. 28) $\hom^C_V(-, -) = \hom_C(-, -)$.

The following propositions are true:

(59) If $\mathrm{Hom}(C) \subseteq V$, then $(\mathrm{hom}_V(a,-))(f) = \langle\langle \mathrm{hom}(a, \mathrm{dom}\, f), \mathrm{hom}(a, \mathrm{cod}\, f)\rangle, \mathrm{hom}(a, f)\rangle$.

(60) If $\mathrm{Hom}(C) \subseteq V$, then $(\mathrm{Obj}(\mathrm{hom}_V(a,-)))(b) = \mathrm{hom}(a,b)$.

(61) If $\mathrm{Hom}(C) \subseteq V$, then $(\mathrm{hom}_V(-,a))(f) = \langle\langle \mathrm{hom}(\mathrm{cod}\, f, a), \mathrm{hom}(\mathrm{dom}\, f, a)\rangle, \mathrm{hom}(f, a)\rangle$.

(62) If $\mathrm{Hom}(C) \subseteq V$, then $(\mathrm{Obj}(\mathrm{hom}_V(-,a)))(b) = \mathrm{hom}(b,a)$.

(63) If $\mathrm{Hom}(C) \subseteq V$, then $(\mathrm{hom}^C_V(-,-))(\langle f^{\mathrm{op}}, g\rangle) = \langle\langle \mathrm{hom}(\mathrm{cod}\, f, \mathrm{dom}\, g), \mathrm{hom}(\mathrm{dom}\, f, \mathrm{cod}\, g)\rangle, \mathrm{hom}(f,g)\rangle$.

(64) If $\mathrm{Hom}(C) \subseteq V$, then $(\mathrm{Obj}(\mathrm{hom}^C_V(-,-)))(\langle a^{\mathrm{op}}, b\rangle) = \mathrm{hom}(a,b)$.

(65) If $\mathrm{Hom}(C) \subseteq V$, then $(\mathrm{hom}^C_V(-,-))(a^{\mathrm{op}},-) = \mathrm{hom}_V(a,-)$.

(66) If $\mathrm{Hom}(C) \subseteq V$, then $(\mathrm{hom}^C_V(-,-))(-,a) = \mathrm{hom}_V(-,a)$.

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