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On the Monoid of Endomorphisms of Universal Algebra and Many Sorted Algebra

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The articles [13], [7], [16], [17], [4], [1], [12], [3], [6], [5], [14], [2], [11], [15], [9], [10], and [8] provide the notation and terminology for this paper.

In this paper U_1 is a universal algebra.

Let us consider U_1. The functor $\mathrm{end}(U_1)$ yielding a non empty set of functions from the carrier of U_1 to the carrier of U_1 is defined by:

(Def. 1) For every function h from U_1 into U_1 holds $h \in \mathrm{end}(U_1)$ iff h is a homomorphism of U_1 into U_1.

Next we state three propositions:

(1) $\mathrm{end}(U_1) \subseteq (\text{the carrier of } U_1)^{\text{the carrier of } U_1}$.

(3)[1] $\mathrm{id}_{\text{the carrier of } U_1} \in \mathrm{end}(U_1)$.

(4) For all elements f_1, f_2 of $\mathrm{end}(U_1)$ holds $f_1 \cdot f_2 \in \mathrm{end}(U_1)$.

Let us consider U_1. The functor $\mathrm{Comp}(U_1)$ yields a binary operation on $\mathrm{end}(U_1)$ and is defined by:

(Def. 2) For all elements x, y of $\mathrm{end}(U_1)$ holds $(\mathrm{Comp}(U_1))(x, y) = y \cdot x$.

Let us consider U_1. The functor $\mathrm{End}(U_1)$ yields a strict multiplicative loop structure and is defined by:

(Def. 3) The carrier of $\mathrm{End}(U_1) = \mathrm{end}(U_1)$ and the multiplication of $\mathrm{End}(U_1) = \mathrm{Comp}(U_1)$ and the unity of $\mathrm{End}(U_1) = \mathrm{id}_{\text{the carrier of } U_1}$.

Let us consider U_1. One can verify that $\mathrm{End}(U_1)$ is non empty.

Let us consider U_1. Observe that $\mathrm{End}(U_1)$ is left unital, well unital, and associative.

We now state two propositions:

(5) For all elements x, y of $\mathrm{End}(U_1)$ and for all elements f, g of $\mathrm{end}(U_1)$ such that $x = f$ and $y = g$ holds $x \cdot y = g \cdot f$.

[1] The proposition (2) has been removed.

(6) $\mathrm{id}_{\text{the carrier of } U_1} = \mathbf{1}_{\mathrm{End}(U_1)}$.

In the sequel S denotes a non void non empty many sorted signature and U_2 denotes a non-empty algebra over S.

Let us consider S, U_2. The functor $\mathrm{end}(U_2)$ yielding a set of many sorted functions from the sorts of U_2 into the sorts of U_2 is defined by the conditions (Def. 4).

(Def. 4)(i) Every element of $\mathrm{end}(U_2)$ is a many sorted function from U_2 into U_2, and

(ii) for every many sorted function h from U_2 into U_2 holds $h \in \mathrm{end}(U_2)$ iff h is a homomorphism of U_2 into U_2.

The following four propositions are true:

(9)[2] $\mathrm{end}(U_2) \subseteq \prod \mathrm{MSFuncs}(\text{the sorts of } U_2, \text{the sorts of } U_2)$.

(10) $\mathrm{id}_{\text{the sorts of } U_2} \in \mathrm{end}(U_2)$.

(11) For all elements f_1, f_2 of $\mathrm{end}(U_2)$ holds $f_1 \circ f_2 \in \mathrm{end}(U_2)$.

(12) For every many sorted function F from $\mathrm{MSAlg}(U_1)$ into $\mathrm{MSAlg}(U_1)$ and for every element f of $\mathrm{end}(U_1)$ such that $F = \{0\} \longmapsto f$ holds $F \in \mathrm{end}(\mathrm{MSAlg}(U_1))$.

Let us consider S, U_2. The functor $\mathrm{Comp}(U_2)$ yielding a binary operation on $\mathrm{end}(U_2)$ is defined by:

(Def. 5) For all elements x, y of $\mathrm{end}(U_2)$ holds $(\mathrm{Comp}(U_2))(x, y) = y \circ x$.

Let us consider S, U_2. The functor $\mathrm{End}(U_2)$ yields a strict multiplicative loop structure and is defined by:

(Def. 6) The carrier of $\mathrm{End}(U_2) = \mathrm{end}(U_2)$ and the multiplication of $\mathrm{End}(U_2) = \mathrm{Comp}(U_2)$ and the unity of $\mathrm{End}(U_2) = \mathrm{id}_{\text{the sorts of } U_2}$.

Let us consider S, U_2. Observe that $\mathrm{End}(U_2)$ is non empty.

Let us consider S, U_2. One can check that $\mathrm{End}(U_2)$ is left unital, well unital, and associative.

We now state three propositions:

(13) For all elements x, y of $\mathrm{End}(U_2)$ and for all elements f, g of $\mathrm{end}(U_2)$ such that $x = f$ and $y = g$ holds $x \cdot y = g \circ f$.

(14) $\mathrm{id}_{\text{the sorts of } U_2} = \mathbf{1}_{\mathrm{End}(U_2)}$.

(16)[3] For every element f of $\mathrm{end}(U_1)$ holds $\{0\} \longmapsto f$ is a many sorted function from $\mathrm{MSAlg}(U_1)$ into $\mathrm{MSAlg}(U_1)$.

Let G, H be non empty groupoids and let I_1 be a map from G into H. We say that I_1 is multiplicative if and only if:

(Def. 7) For all elements x, y of G holds $I_1(x \cdot y) = I_1(x) \cdot I_1(y)$.

Let G, H be non empty multiplicative loop structures and let I_1 be a map from G into H. We say that I_1 is unity-preserving if and only if:

(Def. 8) $I_1(\mathbf{1}_G) = \mathbf{1}_H$.

Let us note that there exists a non empty multiplicative loop structure which is left unital.

Let G, H be left unital non empty multiplicative loop structures. Observe that there exists a map from G into H which is multiplicative and unity-preserving.

Let G, H be left unital non empty multiplicative loop structures. A homomorphism from G to H is a multiplicative unity-preserving map from G into H.

Let G, H be left unital non empty multiplicative loop structures and let h be a map from G into H. We say that h is a monomorphism if and only if:

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[2] The propositions (7) and (8) have been removed.

[3] The proposition (15) has been removed.

(Def. 9) h is one-to-one.

We say that h is an epimorphism if and only if:

(Def. 10) $\operatorname{rng} h =$ the carrier of H.

Let G, H be left unital non empty multiplicative loop structures and let h be a map from G into H. We say that h is an isomorphism if and only if:

(Def. 11) h is an epimorphism and a monomorphism.

The following proposition is true

(17) Let G be a left unital non empty multiplicative loop structure. Then $\mathrm{id}_{\text{the carrier of } G}$ is a homomorphism from G to G.

Let G, H be left unital non empty multiplicative loop structures. We say that G and H are isomorphic if and only if:

(Def. 12) There exists a homomorphism from G to H which is an isomorphism.

Let us note that the predicate G and H are isomorphic is reflexive.

Next we state three propositions:

(18) Let h be a function. Suppose $\operatorname{dom} h = \operatorname{end}(U_1)$ and for every set x such that $x \in \operatorname{end}(U_1)$ holds $h(x) = \{0\} \longmapsto x$. Then h is a homomorphism from $\operatorname{End}(U_1)$ to $\operatorname{End}(\operatorname{MSAlg}(U_1))$.

(19) Let h be a homomorphism from $\operatorname{End}(U_1)$ to $\operatorname{End}(\operatorname{MSAlg}(U_1))$. If for every set x such that $x \in \operatorname{end}(U_1)$ holds $h(x) = \{0\} \longmapsto x$, then h is an isomorphism.

(20) $\operatorname{End}(U_1)$ and $\operatorname{End}(\operatorname{MSAlg}(U_1))$ are isomorphic.

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